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## Combinatorial optimization with 2-joins

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### ABSTRACT

A 2-join is an edge cutset that naturally appears in decomposition of several classes of graphs closed under taking induced sub-graphs, such as perfect graphs and claw-free graphs. In this paper we construct combinatorial polynomial time algorithms for finding a maximum weighted clique, a maximum weighted stable set and an optimal coloring for a class of perfect graphs decomposable by 2-joins: the class of perfect graphs that do not have a balanced skew partition, a 2-join in the complement, nor a homogeneous pair. The techniques we develop are general enough to be easily applied to finding a maximum weighted stable set for another class of graphs known to be decomposable by 2-joins, namely the class of even-hole-free graphs that do not have a star cutset.

We also give a simple class of graphs decomposable by 2-joins into bipartite graphs and line graphs, and for which finding a maximum stable set is NP-hard. This shows that having holes all of the same parity gives essential properties for the use of 2-joins in computing stable sets.

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## 1. Introduction

In this paper all graphs are simple and finite. We say that a graph  $G$  contains a graph  $F$  if  $F$  is isomorphic to an induced subgraph of  $G$ , and it is  $F$ -free if it does not contain  $F$ . A hole in a graph is

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an induced cycle of length at least 4. An *antihole* is the complement of a hole. A graph  $G$  is said to be *perfect* if for every induced subgraph  $G'$  of  $G$ , the chromatic number of  $G'$  is equal to the maximum size of a clique of  $G'$ . A graph is said to be *Berge* if it does not contain an odd hole nor an odd antihole. In 1961, Berge [1] conjectured that every Berge graph is perfect. This was known as the *Strong Perfect Graph Conjecture* (SPGC), it was an object of much research until it was finally proved by Chudnovsky, Robertson, Seymour and Thomas in 2002 [6]. So Berge graphs and perfect graphs are the same class of graphs, but we prefer to write “Berge” for results which rely on the structure of the graphs, and “perfect” for results which rely on the properties of their colorings. We now explain the motivation for this paper and describe informally the results. We use several technical notions that will be defined precisely later.

### 1.1. Optimization with decomposition

In the 1980's, Grötschel, Lovász and Schrijver [22,23] devised a polynomial time algorithm that optimally colors any perfect graph. This algorithm relies on the ellipsoid method and consequently is impractical. Finding a purely combinatorial polynomial time algorithm is still an open question. In fact, after the resolution of the SPGC and the construction of polynomial time recognition algorithm for Berge graphs [5], this is the key open problem in the area.

The proof of the SPGC in [6] was obtained through a *decomposition theorem* for Berge graphs. So, it is a natural question to ask whether this decomposition theorem can be used for coloring and other combinatorial optimization problems. Up to now, it seems that the decomposition theorem is very difficult to use. Let us explain why. In a connected graph  $G$ , a subset of vertices and edges is a *cutset* if its removal disconnects  $G$ . A Decomposition Theorem for a class of graphs  $\mathcal{C}$  is of the following form.

**Decomposition Theorem.** *If  $G$  belongs to  $\mathcal{C}$  then  $G$  is either “basic” or  $G$  has some particular cutset.*

Decomposition theorems can be used for proving theorems. For example, the SPGC was proved using the decomposition theorem for Berge graphs [6], by ensuring that “basic” graphs are simple in the sense that they are easily proved to be perfect directly, and the cutsets used have the property that they cannot occur in a minimum counter-example to the SPGC.

Decomposition theorems can be used also for algorithms. For instance, they yielded many recognition algorithms. To recognize a class  $\mathcal{C}$  with a decomposition theorem, “basic” graphs need to be simple in the sense that they can easily be recognized, and the cutsets used need to have the following property. The removal of a cutset from a graph  $G$  disconnects  $G$  into two or more connected components. From these components *blocks of decomposition* are constructed by adding some more vertices and edges. A decomposition is  $\mathcal{C}$ -*preserving* if it satisfies the following:  $G$  belongs to  $\mathcal{C}$  if and only if all the blocks of decomposition belong to  $\mathcal{C}$ . A recognition algorithm takes a graph  $G$  as input and decomposes it using  $\mathcal{C}$ -preserving decompositions into a polynomial number of basic blocks, which are then checked, in polynomial time, whether they belong to  $\mathcal{C}$ . This is an ideal scenario, and it worked for example for obtaining recognition algorithms for regular matroids (using  $k$ -separations,  $k = 1, 2, 3$ ) [35], max-flow min-cut matroids (using 2-sums and  $\Delta$ -sums) [38], and graphs that do not contain a cycle with a unique chord (using 1-joins and vertex cutsets of size 1 or 2) [37].

But several classes of graphs are too complex for allowing such a direct approach. The main problem is what we call *strong cutsets*. The typical example of a strong cutset is the Chvátal's *star cutset* [7]: a cutset that contains one vertex and a subset of its neighbors. The problem with such a cutset is that it can be very big, for instance, it can be the whole vertex-set except two vertices. And since in the cutset itself, edges are quite unconstrained, knowing that the graph has a star cutset tells little about its structure. From this discussion, it could even be thought that star cutsets are just useless, but this is not the case: deep theorems use strong cutsets. The first one is the Hayward's decomposition theorem of weakly triangulated graphs [24], a simple class of graphs that captures ideas that were used later for all Berge graphs.

More generally, using strong cutsets is essential for proving theorems about many complex classes of graphs closed under taking induced subgraphs, the most famous example being the proof of the

SPGC that uses a generalization of star cutsets: the *balanced skew partition*. Robertson and Seymour have obtained results about minor-closed families of stunning generality, see [28] for a survey. The Robertson–Seymour Theorem [32] states that every minor-closed class of graphs can be characterized by a finite family of excluded minors. Furthermore, every minor-closed property of graphs can be tested in polynomial time [31]. The fact that a unified theory with deep algorithmic consequences exists for classes closed under taking minors and that no such theory exists up to now for the induced subgraph containment relation has perhaps something to do with these strong cutsets.

Yet, for recognition algorithms, strong cutsets can be used. Examples are balanced matrices (using 2-joins and double star cutsets) [12], balanced  $0, \pm 1$  matrices (using 2-joins, 6-joins and double star cutsets) [9], even-hole-free graphs (using 2-joins and star cutsets) [17], and Berge graphs (using 2-joins and double star cutsets from the decomposition theorem in [14]) [5]. This is accomplished by a powerful tool: the *cleaning*, that is a preprocessing of graphs not worth describing here. For combinatorial optimization algorithms (maximum clique, coloring, ...), it seems that the cleaning is useless and no one knows how strong cutsets could be used.

## 1.2. Our results

What we are interested in is whether the known decomposition theorems for perfect graphs [6,4,36] and even-hole-free graphs [11,17] can be used to construct combinatorial polynomial time optimization algorithms. But as we explained above, we do not know how to handle the strong cutsets (namely star cutsets and their generalizations, balanced skew partitions and double star cutsets). So we take the *bottom-up approach*. Let us explain this. In all classes similar to Berge graphs (in the sense that strong cutsets are needed for their decomposition), it can be proved that a decomposition tree can be built by using in a first step only the strong cutsets, and in a second step only the other cutsets (this is not at all obvious for Berge graphs, see [36]). So it is natural to ask whether we can optimize on classes of graphs decomposable by cutsets that are not strong.

For Berge graphs and even-hole-free graphs, if we assume that no strong cutset is needed, we obtain a class of graphs decomposable along 2-joins, a decomposition that was introduced by Cornuéjols and Cunningham in [15] where they prove that no minimum counter-example to the SPGC can admit a 2-join. 2-joins proved to be of great use in decomposition theorems, they were also used in several recognition algorithms mentioned above, but never yet have they been used in construction of optimization algorithms. Proving that a minimally imperfect graph admits no 2-join is done by building blocks of decomposition w.r.t. a 2-join that are smaller graphs with the same clique number and the same chromatic number as the original graph. But as we will see, it is not at all straightforward to transform these ideas into optimization algorithms for our classes.

Our main results are Theorems 9.1 and 9.2. They say that for Berge graphs with no balanced skew partition, no 2-join in the complement and no homogeneous pair, the following problems can be solved combinatorially in polynomial time: maximum weighted clique, maximum weighted stable set and optimal coloring. The homogeneous pair and the 2-join in the complement are not really strong cutsets. Excluding homogeneous pairs was suggested to us by Celina de Figueiredo [18] and is very helpful for several technical reasons, see below. In this bottom-up approach, the next step would be to analyze how homogeneous pairs could be used in optimization algorithms. This step might be doable because some classes of Berge graphs are optimized with homogeneous pairs, see [19]. This might finally lead to a coloring algorithm for Berge graphs with no balanced skew partitions.

Our approach is general enough to give results about even-hole-free graphs that are structurally quite similar to Berge graphs. Their structure was first studied by Conforti, Cornuéjols, Kapoor and Vušković in [11] and [10]. They were focused on showing that even-hole-free graphs can be recognized in polynomial time (a problem that at that time was not even known to be in NP), and their primary motivation was to develop techniques which can then be used in the study of Berge graphs. In [11] a decomposition theorem was obtained using 2-joins, star, double star and triple star cutsets, and in [10] a polynomial time decomposition based recognition algorithm was constructed. Later da Silva and Vušković [17] significantly strengthened the decomposition theorem for even-hole-free graphs by using just 2-joins and star cutsets, which significantly improved the running time of the recognition algorithm for even-hole-free graphs. It is this strengthening that we use in this paper.

One can find a maximum clique in an even-hole-free graph in polynomial time, since as observed by Farber [21] 4-hole-free graphs have  $\mathcal{O}(n^2)$  maximal cliques and hence one can list them all in polynomial time (in all complexity analysis,  $n$  stands for the number of vertices of the input graph and  $m$  for the number of its edges). In [16] da Silva and Vušković show that every even-hole-free graph has a vertex whose neighborhood is hole-free, which leads to a faster algorithm for finding a maximum clique in an even-hole-free graph. The complexities of finding a maximum stable set and an optimal coloring are not known for even-hole-free graphs.

### 1.3. Outline of the paper

In Section 2 we precisely describe all the decomposition theorems we will be working with. For even-hole-graphs, we rely on the theorem of da Silva and Vušković [16]. For the decomposition of Berge graphs we rely on an improvement due to Trotignon [36] of the decomposition theorems of Chudnovsky, Robertson, Seymour and Thomas [6], and Chudnovsky [4]. We need this improvement because we use the so-called non-path 2-joins in the algorithms, and not simply the 2-joins as defined in [6]. For the same reason, we need to exclude the homogeneous pair because some Berge graphs are decomposable only along path 2-join or homogeneous pair (an example is represented Fig. 1).

In Section 3 we show how to construct blocks of decomposition w.r.t. 2-joins that will be class-preserving. This allows us to recursively decompose along 2-joins down to basic graphs.

Using 2-joins in combinatorial optimization algorithms requires building blocks of decomposition and asking at least two questions for at least one block (while for recognition algorithms, one question is enough). When this process is recursively applied it can potentially lead to an exponential blow-up even when the decomposition tree is linear in the size of the input graph. This problem is bypassed by using what we call *extreme 2-joins*, that is 2-joins whose one block of decomposition is basic. In Section 4 we prove that non-basic graphs in our classes actually have extreme 2-joins. Interestingly, we give an example showing that Berge graphs in general do not necessarily have extreme 2-joins, their existence is a special property of graphs with no star cutset. This allows us to build a decomposition tree in which every internal node has two children, at least one of which is a leaf, and hence corresponds to a basic graph.

In Section 5, we show how to put weights on vertices of the block of decomposition w.r.t. an extreme 2-join in order to compute maximum cliques. In fact the approach used here could solve the maximum weighted clique problem for any class with a decomposition theorem along extreme 2-joins down to basic graphs for which the problem can be solved.

For stable sets, the problem is more complicated. As an evidence, in Section 10, we show a simple class of graphs decomposable along extreme 2-joins into bipartite graphs and line graphs of cycles with one chord. This class has a structure close to Berge graphs and in fact much simpler in many respects. Yet, we prove that computing maximum stable sets for this class is NP-hard. So, in Section 6, devoted to stable sets, we need to somehow take advantage of the parity of the cycles. To do so, in Subsection 6.1, we prove a couple of lemmas showing that a maximum weighted stable set and a 2-join overlap in a very special way for graphs where cycles are all of the same parity. These lemmas allow an unusual construction for blocks that preserve simultaneously the weight of a maximal weighted stable set and being Berge.

Our unusual blocks raise some problems. First, if we use them to fully decompose a graph from our class, what we obtain in the leaves of the decomposition tree are not basic graphs, but what we call *extensions* of basic graphs. In Section 7, we show how to solve optimization problems for extensions of basic graphs.

Another problem (that is in fact the source of the previous one) is that our blocks are not class-preserving. They do preserve being Berge, but they introduce balanced skew partitions. To bypass this problem, we construct our decomposition tree in Section 8 in two steps. First, we use classical class-preserving blocks. In the second step, we reprocess the tree to use the unusual blocks.

In Section 9 we give the algorithms for solving the clique and stable set problems. We also recall a classical method to color a perfect graph assuming that subroutines exist for cliques and stable sets. We show that this method can be used for our class.

Section 10 is devoted to the NP-hardness result mentioned above.

**Table 1**  
Classes of graphs.

Class	Definition
$\mathcal{C}^{\text{PARITY}}$	Graphs where all holes have same parity
$\mathcal{C}^{\text{BERGE}}$	Berge graphs
$\mathcal{C}^{\text{OHF}} \mathcal{C}^{\text{EHF}}$	Graphs that do not contain even holes
$\mathcal{C}^{\text{BERGE}}_{\text{NO CUTSET}}$	Berge graphs with no balanced skew partition, no connected non-path 2-join in the complement and no homogeneous pair
$\mathcal{C}^{\text{BERGE}}_{\text{NO BSP}}$	Berge graphs with no balanced skew partition
$\mathcal{C}^{\text{BERGE}}_{\text{BASIC}}$	Bipartite, line graphs of bipartite, path-cobipartite and path-double split graphs; complements of all these graphs
$\mathcal{C}^{\text{EHF}}_{\text{BASIC}}$	Even-hole-free graphs that can be obtained from the line graph of a tree by adding at most two vertices
$\mathcal{C}_{\text{NO SC}}$	Graphs that have no star cutset
$\mathcal{C}^{\text{PARITY}}_{\text{NO SC}}$	Graphs of $\mathcal{C}^{\text{PARITY}}$ that have no star cutset
$\mathcal{C}^{\text{EHF}}_{\text{NO SC}}$	Even-hole-free graphs that have no star cutset

## 2. Decomposition theorems

In this section we introduce all the decomposition theorems we will use in this paper. But before we continue, for the convenience we first establish the following notation for the classes of graphs we will be working with.

We denote by  $\mathcal{C}$  the class of all graphs. We use the superscript  $\text{PARITY}$  to mean that all holes have the same parity. So,  $\mathcal{C}^{\text{PARITY}}$  can be defined equivalently as the union of the odd-hole-free graphs and the even-hole-free graphs. Note that every Berge graph is in  $\mathcal{C}^{\text{PARITY}}$ . We will use the superscript  $\text{EHF}$  to restrict the class to even-hole-free graphs and  $\text{BERGE}$  to restrict the class to Berge graphs. So for instance,  $\mathcal{C}^{\text{BERGE}}$  denotes the class of Berge graphs. We use the subscript  $\text{NO CUTSET}$  to restrict the class to those graphs that do not have a balanced skew partition, a connected non-path 2-join in the complement, nor a homogeneous pair. For technical reasons, mainly to avoid reproving results from [36], we also need the subscript  $\text{NO BSP}$  to restrict a class to graphs with no balanced skew partition. We use the subscript  $\text{NO SC}$  to restrict the class to graphs with no star cutset. We use the subscript  $\text{BASIC}$  to restrict the class to the relevant basic graphs. Table 1 sums up all the classes used in this paper. The classes are defined more formally in the remainder of this section.

We call *path* any connected graph with at least one vertex of degree at most 1 and no vertex of degree greater than 2. A path has at most two vertices of degree 1, which are the *ends* of the path. If  $a, b$  are the ends of a path  $P$  we say that  $P$  is *from a to b*. The other vertices are the *interior* vertices of the path. We denote by  $v_1 - \dots - v_n$  the path whose edge set is  $\{v_1v_2, \dots, v_{n-1}v_n\}$ . When  $P$  is a path, we say that  $P$  is a *path of G* if  $P$  is an induced subgraph of  $G$ . If  $P$  is a path and if  $a, b$  are two vertices of  $P$  then we denote by  $a - P - b$  the only induced subgraph of  $P$  that is path from  $a$  to  $b$ . The *length* of a path is the number of its edges. An *antipath* is the complement of a path. Let  $G$  be a graph and let  $A$  and  $B$  be two subsets of  $V(G)$ . A path of  $G$  is said to be *outgoing from A to B* if it has an end in  $A$ , an end in  $B$ , length at least 2, and no interior vertex in  $A \cup B$ .

The 2-join was first defined by Cornuéjols and Cunningham [15]. A partition  $(X_1, X_2)$  of the vertex-set is a *2-join* if for  $i = 1, 2$ , there exist disjoint non-empty  $A_i, B_i \subseteq X_i$  satisfying the following:

- every vertex of  $A_1$  is adjacent to every vertex of  $A_2$  and every vertex of  $B_1$  is adjacent to every vertex of  $B_2$ ;
- there are no other edges between  $X_1$  and  $X_2$ ;
- for  $i = 1, 2$ ,  $|X_i| \geq 3$ ;
- for  $i = 1, 2$ ,  $X_i$  is not a path of length 2 with an end in  $A_i$ , an end in  $B_i$  and its unique interior vertex in  $C_i = X_i \setminus (A_i \cup B_i)$ .

The sets  $X_1, X_2$  are the two *sides* of the 2-join. When sets  $A_i$ 's and  $B_i$ 's are like in the definition we say that  $(X_1, X_2, A_1, B_1, A_2, B_2)$  is a *split* of  $(X_1, X_2)$ . Implicitly, for  $i = 1, 2$ , we will denote by  $C_i$  the set  $X_i \setminus (A_i \cup B_i)$ .

A 2-join  $(X_1, X_2)$  in a graph  $G$  is said to be *connected* if for  $i = 1, 2$ , there exists a path from  $A_i$  to  $B_i$  with interior in  $C_i$ .

A 2-join is said to be a *path 2-join* if it has a split  $(X_1, X_2, A_1, B_1, A_2, B_2)$  such that for some  $i \in \{1, 2\}$ ,  $G[X_i]$  is a path with an end in  $A_i$ , an end in  $B_i$  and interior in  $C_i$ . Implicitly we will then denote by  $a_i$  the unique vertex in  $A_i$  and by  $b_i$  the unique vertex in  $B_i$ . We say that  $X_i$  is the *path-side* of the 2-join. Note that when  $G$  is not a hole then at most one of  $X_1, X_2$  is a path side of  $(X_1, X_2)$ . A *non-path 2-join* is a 2-join that is not a path 2-join. Note that all the 2-joins used in [9,11,10,12–14] are in fact non-path 2-joins.

### 2.1. Decomposition of even-hole-free graphs

A *vertex cutset* in a graph  $G$  is a set  $S \subset V(G)$  such that  $G \setminus S$  is disconnected ( $G \setminus S$  means  $G[V(G) \setminus S]$ ). By  $N[x]$  we mean  $N(x) \cup \{x\}$ . A *star cutset* in a graph  $G$  is a vertex cutset  $S$  such that for some  $x \in S$ ,  $S \subseteq N[x]$ . Such a vertex  $x$  is called a *center* of the star, and we say that  $S$  is *centered* at  $x$ .

A graph is in  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$  if it is even-hole-free and one can obtain the line graph of a tree by deleting at most two of its vertices.

Building on the work in [25], da Silva and Vušković establish the following strengthening of the original decomposition theorem for even-hole-free graphs [11].

**Theorem 2.1.** (See da Silva and Vušković [17].) *If  $G \in \mathcal{C}^{\text{EHF}}$  then either  $G \in \mathcal{C}_{\text{BASIC}}^{\text{EHF}}$  or  $G$  has a star cutset or a connected non-path 2-join.*

Actually in the decomposition theorem of [17], the basic graphs are defined in a more specific way, but for the purposes of the algorithms the statement of Theorem 2.1 suffices.

### 2.2. Decomposition of Berge graphs

If  $X, Y \subseteq V(G)$  are disjoint, we say that  $X$  is *complete* to  $Y$  if every vertex in  $X$  is adjacent to every vertex in  $Y$ . We also say that  $(X, Y)$  is a *complete pair*. We say that  $X$  is *anticomplete* to  $Y$  if there are no edges between  $X$  and  $Y$ . We also say that  $(X, Y)$  is an *anticomplete pair*. We say that a graph  $G$  is *anticonnected* if its complement  $\bar{G}$  is connected.

Skew partitions were first introduced by Chvátal [7]. A *skew partition* of a graph  $G = (V, E)$  is a partition of  $V$  into two sets  $A$  and  $B$  such that  $A$  induces a graph that is not connected, and  $B$  induces a graph that is not anticonnected. When  $A_1, A_2, B_1, B_2$  are non-empty sets such that  $(A_1, A_2)$  partitions  $A$ ,  $(A_1, A_2)$  is an anticomplete pair,  $(B_1, B_2)$  partitions  $B$ , and  $(B_1, B_2)$  is a complete pair, we say that  $(A_1, A_2, B_1, B_2)$  is a *split* of the skew partition  $(A, B)$ . A *balanced skew partition* (first defined in [6]) is a skew partition  $(A, B)$  with the additional property that every induced path of length at least 2 with ends in  $B$ , interior in  $A$  has even length, and every antipath of length at least 2 with ends in  $A$ , interior in  $B$  has even length. If  $(A, B)$  is a skew partition, we say that  $B$  is a *skew cutset*. If  $(A, B)$  is balanced we say that the skew cutset  $B$  is *balanced*. Note that Chudnovsky et al. [6] proved that no minimum counter-example to the strong perfect graph conjecture admits a balanced skew partition.

Call *double split graph* (first defined in [6]) any graph  $G$  that may be constructed as follows. Let  $k, l \geq 2$  be integers. Let  $A = \{a_1, \dots, a_k\}$ ,  $B = \{b_1, \dots, b_k\}$ ,  $C = \{c_1, \dots, c_l\}$ ,  $D = \{d_1, \dots, d_l\}$  be four disjoint sets. Let  $G$  have vertex-set  $A \cup B \cup C \cup D$  and edges in such a way that:

- $a_i$  is adjacent to  $b_i$  for  $1 \leq i \leq k$ . There are no edges between  $\{a_i, b_i\}$  and  $\{a_{i'}, b_{i'}\}$  for  $1 \leq i < i' \leq k$ .
- $c_j$  is non-adjacent to  $d_j$  for  $1 \leq j \leq l$ . There are all four edges between  $\{c_j, d_j\}$  and  $\{c_{j'}, d_{j'}\}$  for  $1 \leq j < j' \leq l$ .
- There are exactly two edges between  $\{a_i, b_i\}$  and  $\{c_j, d_j\}$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq l$  and these two edges are disjoint.

The homogeneous pair was first defined by Chvátal and Sbihi [8]. The definition that we give here is a slight variation. A *homogeneous pair* is a partition of  $V(G)$  into six sets  $(A, B, C, D, E, F)$  such that:

- $A, B, C, D$  and  $F$  are non-empty (but  $E$  is possibly empty);
- every vertex in  $A$  has a neighbor in  $B$  and a non-neighbor in  $B$ , and vice versa (note that this implies that  $A$  and  $B$  both contain at least 2 vertices);
- the pairs  $(C, A)$ ,  $(A, F)$ ,  $(F, B)$ ,  $(B, D)$  are complete;
- the pairs  $(D, A)$ ,  $(A, E)$ ,  $(E, B)$ ,  $(B, C)$  are anticomplete.

All the decomposition theorems for Berge graphs that we mention now are published in papers that have a definition of a connected 2-join and a homogeneous pair slightly more restrictive than ours. So, the statements that we give here follow directly from the original statements.

The following theorem was first conjectured in a slightly different form by Conforti, Cornuéjols and Vušković, who proved it in the particular case of square-free graphs [13]. A corollary of it is the Strong Perfect Graph Theorem.

**Theorem 2.2.** (See Chudnovsky, Robertson, Seymour and Thomas [6].) *Let  $G$  be a Berge graph. Then either  $G$  is bipartite, line graph of bipartite, complement of bipartite, complement of line graph of bipartite or double split, or  $G$  has a homogeneous pair, or  $G$  has a balanced skew partition or one of  $G, \bar{G}$  has a connected 2-join.*

The theorem that we state now is due to Chudnovsky who proved it from scratch, that is without assuming Theorem 2.2. Her proof uses the notion of *trigraph*. The theorem shows that homogeneous pairs are not necessary to decompose Berge graphs. Thus it is a result stronger than Theorem 2.2.

**Theorem 2.3.** (See Chudnovsky [4,3].) *Let  $G$  be a Berge graph. Then either  $G$  is bipartite, line graph of bipartite, complement of bipartite, complement of line graph of bipartite or double split, or one of  $G, \bar{G}$  has a connected 2-join or  $G$  has a balanced skew partition.*

### 2.3. Avoiding path 2-joins in Berge graphs

Theorem 2.3 allows path 2-joins and they are not easy to handle. Because their path side is sometimes not substantial enough to allow building a block of decomposition that carries sufficiently information. But there are other reasons that we explain now. Let us first remind the starting point of this work: we do not know how to handle skew partitions in algorithms. So, things should be easier for a Berge graph with no skew partition. Such a graph is likely to have a 2-join, but when decomposing along this 2-join, we may create a skew partition again. Thus, it seems impossible to devise a recursive algorithm that decomposes graphs with no balanced skew partitions along 2-joins. A careful study of this phenomenon, done in [36], shows that path 2-joins, more precisely certain kinds of path 2-join, are responsible for this and can be avoided. Let us state this more precisely.

The following theorem shows that path 2-joins are not necessary to decompose Berge graphs, but at the expense of extending balanced skew partitions to general skew partitions and introducing a new basic class. So, this theorem is useless for us (at least, we do not know how to use it). Before stating the theorem, we need to define the new basic class.

A graph  $G$  is *path-cobipartite* if it is a Berge graph obtained by subdividing an edge between the two cliques that partitions the complement of a bipartite graph. More precisely, a graph is *path-cobipartite* if its vertex-set can be partitioned into three sets  $A, B, P$  where  $A$  and  $B$  are non-empty cliques and  $P$  consist of vertices of degree 2, each of which belongs to the interior of a unique path of odd length with one end  $a$  in  $A$ , the other one  $b$  in  $B$ . Moreover,  $a$  has neighbors only in  $A \cup P$  and  $b$  has neighbors only in  $B \cup P$ . Note that a path-cobipartite graph such that  $P$  is empty is the complement of bipartite graph. Note that our path-cobipartite graphs are simply the complement of the *path-bipartite* graphs defined by Chudnovsky in [3]. For convenience, we prefer to think about them in the complement as we do.

**Theorem 2.4.** (See Chudnovsky [3].) *Let  $G$  be a Berge graph. Then either  $G$  is bipartite, line graph of bipartite, complement of bipartite, complement of line graph of bipartite, double split, path-bipartite, complement of path-bipartite, or  $G$  has a connected non-path 2-join, or  $\bar{G}$  has a connected 2-join, or  $G$  has a homogeneous pair or  $G$  has a skew partition.*

A *path-double split graph* is any graph  $H$  that may be constructed as follows. Let  $k, l \geq 2$  be integers. Let  $A = \{a_1, \dots, a_k\}$ ,  $B = \{b_1, \dots, b_k\}$ ,  $C = \{c_1, \dots, c_l\}$ ,  $D = \{d_1, \dots, d_l\}$  be four disjoint sets. Let  $E$  be another possibly empty set disjoint from  $A, B, C, D$ . Let  $H$  have vertex-set  $A \cup B \cup C \cup D \cup E$  and edges in such a way that:

- For every vertex  $v$  in  $E$ ,  $v$  has degree 2 and there exists  $i \in \{1, \dots, k\}$  such that  $v$  lies on a path of odd length from  $a_i$  to  $b_i$ .
- For  $1 \leq i \leq k$ , there is a unique path of odd length (possibly 1) between  $a_i$  and  $b_i$  whose interior is in  $E$ . There are no edges between  $\{a_i, b_i\}$  and  $\{a_{i'}, b_{i'}\}$  for  $1 \leq i < i' \leq k$ .
- $c_j$  is non-adjacent to  $d_j$  for  $1 \leq j \leq l$ . There are all four edges between  $\{c_j, d_j\}$  and  $\{c_{j'}, d_{j'}\}$  for  $1 \leq j < j' \leq l$ .
- There are exactly two edges between  $\{a_i, b_i\}$  and  $\{c_j, d_j\}$  for  $1 \leq i \leq k$ ,  $1 \leq j \leq l$  and these two edges are disjoint.

Note that a path-double split graph  $G$  has an obvious skew partition that is not balanced:  $(A \cup B \cup E, C \cup D)$ . In fact, it is proved in [36], Lemma 4.5, that this is the unique skew partition of  $G$ . Also, either  $E$  is empty and the graph is a double split graph or  $E$  is not empty and the graph has a path 2-join. Path-double split graphs are the reason why in Theorem 2.4, one needs to add non-balanced skew partitions in the list of decompositions.

We call *flat path* of a graph  $G$  any path of length at least 2, whose interior vertices all have degree 2 in  $G$  and whose ends have no common neighbors outside the path. A *homogeneous 2-join* is a partition of  $V(G)$  into six non-empty sets  $(A, B, C, D, E, F)$  such that:

- $(A, B, C, D, E, F)$  is a homogeneous pair such that  $E$  is not empty;
- every vertex in  $E$  has degree 2 and belongs to a flat path of odd length with an end in  $C$ , an end in  $D$  and whose interior is in  $E$ ;
- every flat path outgoing from  $C$  to  $D$  and whose interior is in  $E$  is the path-side of a non-cutting connected 2-join of  $G$ .

Note we have not defined *cutting* and *non-cutting* 2-joins. The definition is long (see [36]) and we do not need it here because the only property of homogeneous 2-joins that we are going to use is that they imply the existence of a homogeneous pair. Homogeneous 2-joins are the reason why in Theorem 2.4, one needs to add homogeneous pairs in the list of decompositions.

The following theorem generalizes the previously known decomposition theorems for Berge graphs. So it implies the Strong Perfect Graph Theorem, but its proof relies heavily on Theorem 2.3. Hence it does not give a new proof of the Strong Perfect Graph Theorem.

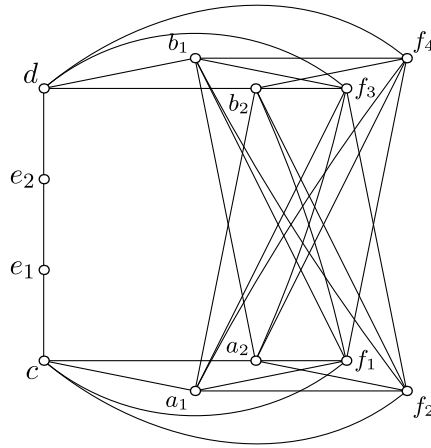
**Theorem 2.5.** (See Trotignon [36].) *Let  $G$  be a Berge graph. Then either  $G$  is bipartite, line graph of bipartite, complement of bipartite, complement of line graph of bipartite or double split, or one of  $G, \bar{G}$  is a path-cobipartite graph, or one of  $G, \bar{G}$  is a path-double split graph, or one of  $G, \bar{G}$  has a homogeneous 2-join, or one of  $G, \bar{G}$  has a connected non-path 2-join, or  $G$  has a balanced skew partition.*

Here, we will only use the obvious following corollary:

**Theorem 2.6.** *If  $G$  is in  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$ , then either  $G$  is in  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$  or  $G$  has a connected non-path 2-join.*

**Proof.** Follows directly from Theorem 2.5 and the fact that a graph with a homogeneous 2-join, or whose complement has a homogeneous 2-join, admits a homogeneous pair.  $\square$





**Fig. 1.** A graph that has a homogeneous 2-join  $(\{a_1, a_2\}, \{b_1, b_2\}, \{c\}, \{d\}, \{e_1, e_2\}, \{f_1, f_2, f_3, f_4\})$ .

Note that since we need to use non-path 2-joins, we really need to exclude homogeneous pairs (or to find a different approach). Indeed, there exist Berge graphs that are decomposable only with path 2-joins and homogeneous pairs. An example from [36] is shown Fig. 1.

### 3. Blocks of decomposition with respect to a 2-join

Blocks of decomposition with respect to a 2-join are built by replacing each side of the 2-join by a path and the lemma below shows that for graphs in  $\mathcal{C}^{\text{PARITY}}$  there exists a unique way to choose the parity of that path.

**Lemma 3.1.** *Let  $G$  be a graph in  $\mathcal{C}^{\text{PARITY}}$  and  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a connected 2-join of  $G$ . Then for  $i = 1, 2$ , all the paths with an end in  $A_i$ , an end in  $B_i$  and interior in  $C_i$  have the same parity.*

**Proof.** Since  $(X_1, X_2)$  is connected there exists a path  $P$  with one end in  $A_{3-i}$ , one end in  $B_{3-i}$  and interior in  $C_{3-i}$ . If two paths  $Q, R$  from  $A_i$  to  $B_i$  with interior in  $C_i$  are of different parity then the holes  $P \cup Q$  and  $P \cup R$  are of different parity, a contradiction to  $G \in \mathcal{C}^{\text{PARITY}}$ .  $\square$

Let  $G$  be a graph and  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a connected 2-join of  $G$ . Let  $k_1, k_2 \geq 1$  be integers. The *blocks of decomposition* of  $G$  with respect to  $(X_1, X_2)$  are the two graphs  $G_1^{k_1}, G_2^{k_2}$  that we describe now. We obtain  $G_1^{k_1}$  by replacing  $X_2$  by a *marker path*  $P_2$ , of length  $k_1$ , from a vertex  $a_2$  complete to  $A_1$ , to a vertex  $b_2$  complete to  $B_1$  (the interior of  $P_2$  has no neighbor in  $X_1$ ). The block  $G_2^{k_2}$  is obtained similarly by replacing  $X_1$  by a marker path  $P_1$  of length  $k_2$ . We say that  $G_1^{k_1}$  and  $G_2^{k_2}$  are *parity-preserving* if  $G$  is in  $\mathcal{C}^{\text{PARITY}}$ , for  $i = 1, 2$  and for a path  $Q_i$  from  $A_i$  to  $B_i$  whose intermediate vertices are in  $C_i$  (and such a path exists since  $(X_1, X_2)$  is connected), the marker path  $P_i$  has the same parity as  $Q_i$ . Note that by Lemma 3.1, our definition does not depend on the choice of a particular  $Q_i$ .

#### 3.1. Interaction between 2-joins and cutsets

Here we show that assuming that a graph does not admit the cutsets we consider gives several interesting properties for its 2-joins.

**Lemma 3.2.** *Let  $G$  be in  $\mathcal{C}_{\text{NO SC}}$  and let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a 2-join of  $G$ . Then the following hold:*

- (i) every component of  $G[X_i]$  meets both  $A_i$  and  $B_i$ ,  $i = 1, 2$ ;
- (ii)  $(X_1, X_2)$  is connected;
- (iii) every  $u \in X_i$  has a neighbor in  $X_i$ ,  $i = 1, 2$ ;
- (iv) every vertex of  $A_i$  has a non-neighbor in  $B_i$ ,  $i = 1, 2$ ;
- (v) every vertex of  $B_i$  has a non-neighbor in  $A_i$ ,  $i = 1, 2$ ;
- (vi)  $|X_i| \geq 4$ ,  $i = 1, 2$ .

**Proof.** To prove (i), suppose for a contradiction that some connected component  $C$  of  $G[X_1]$  does not intersect  $B_1$  (the other cases are symmetric). If there is a vertex  $c \in C \setminus A_1$  then for any vertex  $u \in A_2$ , we have that  $\{u\} \cup A_1$  is a star cutset that separates  $c$  from  $B_1$ . So,  $C \subseteq A_1$ . If  $|A_1| \geq 2$  then pick any vertex  $c \in C$  and a vertex  $c' \neq c$  in  $A_1$ . Then  $\{c'\} \cup A_2$  is a star cutset that separates  $c$  from  $B_1$ . So,  $C = A_1 = \{c\}$ . Hence, there exists some component of  $G[X_1]$  that does not intersect  $A_1$ , so by the same argument as above we deduce  $|B_1| = 1$  and the unique vertex of  $B_1$  has no neighbor in  $X_1$ . Since  $|X_1| \geq 3$ , there is a vertex  $u$  in  $C_1$ . For any vertex  $v$  in  $X_2$ ,  $\{v\}$  is a star cutset of  $G$  that separates  $u$  from  $A_1$ , a contradiction.

Item (ii) follows directly from (i).

To prove (iii), just notice that if some vertex in  $X_i$  has no neighbor in  $X_i$ , then it forms a component of  $G[X_i]$  that does not intersect one of  $A_i, B_i$ . This is a contradiction to (i).

To prove (iv) and (v), consider a vertex  $a \in A_1$  complete to  $B_1$  (the others cases are symmetric). If  $A_1 \cup C_1 \neq \{a\}$  then  $B_1 \cup A_2 \cup \{a\}$  is a star cutset that separates  $(A_1 \cup C_1) \setminus \{a\}$  from  $B_2$ , a contradiction. So,  $A_1 \cup C_1 = \{a\}$  and  $|B_1| \geq 2$  because  $X_1 \geq 3$ . Let  $b \neq b' \in B_1$ . So,  $\{b, a\} \cup B_2$  is a star cutset that separates  $b'$  from  $A_2$ , a contradiction.

To prove (vi), suppose for a contradiction that  $|X_1| = 3$ . Up to symmetry we assume  $|A_1| = 1$ , and let  $a_1$  be the unique vertex in  $A_1$ . As we just proved, every vertex of  $B_1$  has a non-neighbor in  $A_1$ . Since  $A_1 = \{a_1\}$ , this means that  $a_1$  has no neighbor in  $B_1$ . Since  $(X_1, X_2)$  is a connected 2-join (because of (ii)),  $G[X_1]$  must be path of length 2 whose interior is in  $C_1$ . This contradicts the definition of a 2-join.  $\square$

Note that a star cutset of size at least 2 is a skew cutset. The following was noticed by Zambelli [39] and is sometimes very useful. A proof can be found in [36], Lemma 4.3.

**Lemma 3.3.** *Let  $G$  be a Berge graph of order at least 5 with at least one edge. If  $G$  has a star cutset then  $G$  has a balanced skew partition.*

**Lemma 3.4.** *If a graph  $G$  is in  $\mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$  and admits a 2-join or a homogeneous pair, then neither  $G$  nor  $\bar{G}$  has a star cutset.*

**Proof.** Assume the hypothesis. Since  $G$  has 2-join or a homogeneous pair, it is of order at least 5 and both  $G$  and  $\bar{G}$  have at least one edge. So, by Lemma 3.3, if  $G$  has a star cutset, then  $G$  has a balanced skew partition, a contradiction. Similarly, by Lemma 3.3, if  $\bar{G}$  has a star cutset, then  $\bar{G}$  has a balanced skew partition, and hence so does  $G$ , a contradiction.  $\square$

**Lemma 3.5.** *Let  $G \in \mathcal{C}_{\text{NO SC}}^{\text{EHF}} \cup \mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$ . If  $(X_1, X_2, A_1, B_1, A_2, B_2)$  is a split of a 2-join of  $G$  then every vertex of  $A_i$  has a neighbor in  $X_i \setminus A_i$ ,  $i = 1, 2$ ; and every vertex of  $B_i$  has a neighbor in  $X_i \setminus B_i$ ,  $i = 1, 2$ .*

**Proof.** By Lemma 3.4,  $G$  has no star cutset, so by Lemma 3.2(ii),  $(X_1, X_2)$  is connected. Consider a vertex  $a \in A_1$  with no neighbor in  $X_1 \setminus A_1$  (the other cases are similar). We put  $Z = (A_1 \cup A_2) \setminus \{a\}$ . So,  $Z$  is a cutset that separates  $a$  from the rest of the graph. We note that  $A_1 \neq \{a\}$  because  $(X_1, X_2)$  is connected. If  $G \in \mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$  then  $V(G) \setminus Z$  is a star cutset (centered at  $a$ ) of  $\bar{G}$  and this contradicts Lemma 3.4. If  $G \in \mathcal{C}_{\text{NO SC}}^{\text{EHF}}$  then we note that at least one of  $A_1, A_2$  is a clique because  $G$  contains no 4-hole. So  $Z$  is a star cutset of  $G$ , a contradiction.  $\square$

### 3.2. Staying in the class

Here we prove several lemmas of the same flavor, needed later for inductive proofs and recursive algorithms. They all say that building the blocks of a graph with respect to some well chosen 2-join preserves several properties like being free of cutset or member of some class.

**Lemma 3.6.** *Let  $G$  be a graph in  $\mathcal{C}_{\text{NO SC}}^{\text{PARITY}}$  and  $(X_1, X_2)$  a connected 2-join of  $G$ . Let  $G_1^{k_1}, G_2^{k_2}$  be parity-preserving blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$ , where  $k_1, k_2 \geq 2$ . If one of  $G_1^{k_1}, G_2^{k_2}$  contain an odd (resp. even) hole, then  $G$  contains an odd (resp. even) hole.*

**Proof.** Let  $C$  be a hole in  $G_1^{k_1}$  say. Let  $P_2 = a_2 - \dots - b_2$  be the marker path of  $G_1^{k_1}$ . If  $C \subseteq X_1 \cup \{a_2, b_2\}$  then we obtain a hole  $C'$  of  $G$  as follows. By Lemma 3.2(iv), there exist non-adjacent vertices  $a'_2 \in A_2, b'_2 \in B_2$ . If  $a_2 \in C$  (resp. if  $b_2 \in C$ ) then we replace  $a_2$  (resp.  $b_2$ ) by  $a'_2$  (resp.  $b'_2$ ). So, holes  $C$  and  $C'$  have the same length and in particular the same parity.

So we may assume that  $C$  contains interior vertices of  $P_2$ . This means that  $C$  is the union of  $P_2$  together with a path  $Q$  from  $A_1$  to  $B_1$  with interior in  $C_1$ . We obtain a hole  $C'$  of  $G$  by replacing  $P_2$  by any path of  $G$  from  $A_2$  to  $B_2$  with interior in  $C_2$ . Such a path exists because  $(X_1, X_2)$  is connected. Holes  $C$  and  $C'$  have the same parity from the definition of parity-preserving blocks and Lemma 3.1.  $\square$

**Lemma 3.7.** *Let  $G \in \mathcal{C}_{\text{NO SC}}^{\text{EHF}} \cup \mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$  and let  $(X_1, X_2)$  be a connected 2-join of  $G$ . Let  $G_1^{k_1}$  and  $G_2^{k_2}$  be blocks of decomposition w.r.t.  $(X_1, X_2)$ . If  $k_1, k_2 \geq 3$  then  $G_1^{k_1}$  and  $G_2^{k_2}$  are both in  $\mathcal{C}_{\text{NO SC}}$ .*

**Proof.** By Lemma 3.4,  $G$  is in  $\mathcal{C}_{\text{NO SC}}$ . Let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of  $(X_1, X_2)$ . Let  $P_2 = a_2 - \dots - b_2$  be the marker path of  $G_1^{k_1}$ .

Suppose that  $G_1^{k_1}$  say has a star cutset  $S$  centered at  $x$ . If  $S \cap P_2 = \emptyset$  then  $S$  is a star cutset of  $G$ , a contradiction. So  $S \cap P_2 \neq \emptyset$ . If  $x \notin P_2$  then w.l.o.g.  $x \in A_1$  and hence  $S \cup A_2$  is a star cutset of  $G$ , a contradiction.

So  $x \in P_2$ . First suppose that  $x$  coincides with  $a_2$  or  $b_2$ , say  $x = a_2$ . Since  $k_1 > 1$ , vertices of  $B_1 \cup \{b_2\}$  are all contained in the same component  $B$  of  $G_1^{k_1} \setminus S$ . Let  $C$  be a connected component of  $G_1^{k_1} \setminus S$  that is distinct from  $B$ . If  $C \cap A_1 \neq \emptyset$  then for  $a'_2 \in A_2, A_1 \cup \{a'_2\}$  is a star cutset of  $G$ , a contradiction. So  $C \subseteq A_1$ . Hence, some vertex  $c$  of  $C$  is in  $A_1$  and has no neighbor in  $X_1 \setminus A_1$ , a contradiction to Lemma 3.5.

Therefore,  $x \in P_2 \setminus \{a_2, b_2\}$ . Since  $(X_1, X_2)$  satisfies (i) from Lemma 3.2, both  $G[X_1 \cup \{a_2\}]$  and  $G[X_1 \cup \{b_2\}]$  are connected. So, both  $a_2$  and  $b_2$  must be in  $S$ , a contradiction to  $k_1 \geq 3$ .  $\square$

**Lemma 3.8.** *Let  $G \in \mathcal{C}_{\text{NO SC}}^{\text{EHF}}$ . Let  $(X_1, X_2)$  be a connected 2-join of  $G$  and let  $G_1^{k_1}$  and  $G_2^{k_2}$  be parity-preserving blocks of decomposition. If  $k_1, k_2 \geq 3$  then  $G_1^{k_1}$  and  $G_2^{k_2}$  are both in  $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$ .*

**Proof.** For  $i = 1, 2$ , by Lemma 3.7,  $G_i^{k_i}$  has no star cutset. By Lemma 3.6,  $G_i^{k_i}$  contains no even hole.  $\square$

**Lemma 3.9.** *Let  $G \in \mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$  and let  $(X_1, X_2)$  be a connected non-path 2-join of  $G$ . Let  $G_1^{k_1}$  and  $G_2^{k_2}$  be parity-preserving blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$  where  $3 \leq k_1, k_2 \leq 4$ . Then  $G_1^{k_1}$  and  $G_2^{k_2}$  are both in  $\mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$ .*

**Proof.** Lemmas 4.12 and 4.18 from [36] say that if  $(X_1, X_2)$  is proper then the conclusion holds, where a proper 2-join means a 2-join that satisfies (i) from Lemma 3.2. So, by Lemma 3.2,  $(X_1, X_2)$  is proper and the conclusion holds.  $\square$

**Lemma 3.10.** *Let  $G \in \mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$  and let  $(A, B, C, D, E, F)$  be a homogeneous pair of  $G$ . Then every vertex of  $C$  has a neighbor in  $E \cup D$  and every vertex of  $D$  has a neighbor in  $E \cup C$ .*

**Proof.** If there exists a vertex  $c \in C$  with no neighbor in  $E \cup D$  then  $(A \cup C \cup F) \setminus \{c\}$  is a skew cutset that separates  $c$  from the rest of the graph. Hence,  $\overline{G}$  has a star-cutset centered at  $c$ , a contradiction to Lemma 3.4. The case with  $d \in D$  is similar.  $\square$

Berge graphs have a particular problem: their decomposition theorems allow 2-joins in the complement. And swapping to the complement makes difficult keeping track of maximum stable sets and cliques. The following lemma shows how to bypass this problem.

**Lemma 3.11.** *Let  $G \in \mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$  and let  $(X_1, X_2)$  be a connected non-path 2-join of  $G$ . Let  $G_1^{k_1}$  and  $G_2^{k_2}$  be parity-preserving blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$  where  $3 \leq k_1, k_2 \leq 4$ . Then  $G_1^{k_1}$  and  $G_2^{k_2}$  are in  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$ .*

**Proof.** Note that  $G$  is in  $\mathcal{C}_{\text{NO BSP}}^{\text{BERGE}}$ . By Lemma 3.9,  $G_1^{k_1}$  and  $G_2^{k_2}$  are both Berge graphs and  $G_1^{k_1}$  and  $G_2^{k_2}$  have no balanced skew partition. Because of the symmetry, we just have to prove the following two claims:

- (1)  $\overline{G_2^{k_2}}$  has no connected 2-join.

**Proof of (1).** Note that  $G_2^{k_2}$  has a 2-join (a path 2-join). So by Lemma 3.4,  $\overline{G_2^{k_2}}$  has no star cutset. In  $G_2^{k_2}$ , we denote by  $P_1 = a_1 - c_1 - c_2 - c' - b_1$  the marker path, where  $a_1$  is complete to  $A_2$  and  $b_1$  to  $B_2$ . Note that this path may be of length 3 or 4. If it is of length 3, then we suppose  $c' = b_1$ , this is convenient to avoid a multiplication of cases. Note that by Lemma 3.2(ii), any 2-join of  $\overline{G_2^{k_2}}$  is connected. Let us suppose for a contradiction that  $\overline{G_2^{k_2}}$  has a 2-join. Let  $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$  be a split of this 2-join. We put  $C'_i = X'_i \setminus (A'_i \cup B'_i)$ ,  $i = 1, 2$ .

Since  $c_1, c_2$  are of degree 2 in  $G_2^{k_2}$ , they are of degree  $|V(\overline{G_2^{k_2}})| - 3$  in  $\overline{G_2^{k_2}}$ . Since by Lemma 3.2(vi),  $|X'_i| \geq 4$ ,  $i = 1, 2$ , we must have  $c_1, c_2 \in A'_1 \cup A'_2 \cup B'_1 \cup B'_2$ . Also,  $c_1, c_2$  are non-adjacent in  $\overline{G_2^{k_2}}$  so up to symmetry there are two cases. Here below, the words “neighbor” and “non-neighbor” refer to adjacency in  $\overline{G_2^{k_2}}$ .

Case 1:  $c_1 \in A'_1$ ,  $c_2 \in B'_2$ . Then by Lemma 3.2(iv) applied to  $(X'_1, X'_2)$ ,  $c_1$  must have a non-neighbor in  $B'_1$ , and this non-neighbor must be  $a_1$  ( $a_1, c_2$  are the only non-neighbors of  $c_1$ ). So,  $a_1 \in B'_1$ . Similarly,  $c_2$  must have a non-neighbor in  $A'_2$ , and this non-neighbor must be  $c'$ , i.e.  $c' \in A'_2$ . But then, since  $a_1 c'$  is an edge of  $\overline{G_2^{k_2}}$ , this contradicts the definition of a 2-join.

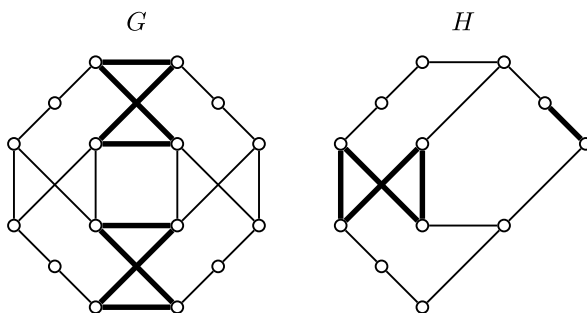
Case 2:  $c_1 \in A'_1$ ,  $c_2 \in B'_1$ . We must have  $B'_2 = \{a_1\}$  because  $a_1, c_2$  are the only non-neighbors of  $c_1$ . For the same reason,  $A'_2 = \{c'\}$ . Since  $a_1$  is a neighbor of  $c'$ , there is a contradiction to Lemma 3.2(iv) applied to  $(X'_1, X'_2)$ . This proves (1).

- (2)  $G_2^{k_2}$  has no homogeneous pair.

**Proof of (2).** Suppose for a contradiction that  $(A, B, C, D, E, F)$  is a homogeneous pair of  $G_2^{k_2}$ . It is convenient to use now a slightly different notation for the marker path  $P_1 = a_1 - c_1 - c_2 - c' - b_1$ . Now, when the path is of length 3, we suppose  $c' = c_2$ .

By Lemma 3.10, every vertex in  $A, B, C, D, F$  has degree at least 3. So,  $c_1, c_2, c' \in E$ . We will reach the contradiction by giving a homogeneous pair  $(A', B', C', D', E', F')$  of  $G$ .

In  $E'$ , we put every vertex of  $E \setminus \{c_1, c_2, c'\}$ , and we add  $C_1$ . If  $E$  contains  $a_1$ , we add  $A_1$  to  $E'$ . If  $E$  contains  $b_1$ , we add  $B_1$  to  $E'$ . We set  $A' = A$  and  $B' = B$ . In  $C'$ , we put every vertex of  $C \setminus \{a_1, b_1\}$ . If  $C$  contains  $a_1$ , we add  $A_1$  to  $C'$ . If  $C$  contains  $b_1$ , we add  $B_1$  to  $C'$ . We define similarly  $D', F'$  from  $D$  and  $F$  respectively. Now we observe that  $(A', B', C', D', E', F')$  partitions  $V(G)$  and is a homogeneous pair of  $G$  (note that  $E'$  is possibly empty). This proves (2).  $\square$



**Fig. 2.** Graph  $G$  (on the left) has a star cutset, but does not have an extreme connected non-path 2-join:  $G$  has a connected non-path 2-join represented with bold lines, and all connected non-path 2-joins are equivalent to this one. Both of the blocks of decomposition are isomorphic to graph  $H$  (on the right), and  $H$  has a connected non-path 2-join whose edges are represented with bold lines.

#### 4. Extreme 2-joins

In this section, we show how to find a 2-join in a graph so that one of the blocks has no more 2-joins. The idea behind this is that when  $(X_1, X_2)$  is a 2-join of  $G$  and the block of decomposition  $G_1^{k_1}$  has no more decomposition then it is basic. So a lot of computations can be made on  $G[X_1]$  without leading to an exponential complexity.

Let  $(X_1, X_2)$  be a connected non-path 2-join of a graph  $G$ . We say that  $(X_1, X_2)$  is a *minimally-sided connected non-path 2-join* if for some  $i \in \{1, 2\}$ , the following holds: for every connected non-path 2-join  $(X'_1, X'_2)$  of  $G$ , neither  $X'_1 \subsetneq X_i$  nor  $X'_2 \subsetneq X_i$  holds. We call  $X_i$  a *minimal side* of this minimally-sided 2-join. Note that minimally-sided connected non-path 2-joins exist in any graph that admits a connected non-path 2-join.

Let  $(X_1, X_2)$  be a connected non-path 2-join of  $G$ . We say that  $(X_1, X_2)$  is an *extreme 2-join* if for some  $i \in \{1, 2\}$  and all  $k \geq 3$  the block of decomposition  $G_i^k$  has no connected non-path 2-join. We say that  $X_i$  is an *extreme side* of such a 2-join. Fig. 2 shows that graphs in general do not have an extreme 2-join, but as we now show, graphs with no star cutset do.

**Lemma 4.1.** *Let  $G \in \mathcal{C}_{\text{No SC}}$  and let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a connected non-path 2-join of  $G$ . If  $|A_1| = 1$  then  $(X_1 \setminus A_1, X_2 \cup A_1, N_{G[X_1]}(A_1), B_1, A_1, B_2)$  is a split of a connected non-path 2-join of  $G$ .*

**Proof.** Assume  $A_1 = \{a_1\}$ . So by Lemma 3.2(v),  $a_1$  has no neighbor in  $B_1$ . Let  $A'_1 = N_{G[X_1]}(a_1)$ . Then  $A'_1 \cap B_1 = \emptyset$ . By Lemma 3.2(vi),  $|X_1 \setminus A_1| \geq 3$ . Since  $(X_1, X_2)$  is a non-path 2-join of  $G$ , it follows that  $(X_1 \setminus A_1, X_2 \cup A_1, A'_1, B_1, A_1, B_2)$  is a split of a non-path 2-join of  $G$ . By Lemma 3.2(ii),  $(X_1 \setminus A_1, X_2 \cup A_1)$  is connected.  $\square$

**Lemma 4.2.** *Let  $G \in \mathcal{C}_{\text{No SC}}$  and let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a minimally-sided connected non-path 2-join of  $G$ . Let  $X_1$  be a minimal side. Then  $|A_1| \geq 2$  and  $|B_1| \geq 2$ . In particular,  $A_2 \cup B_2$  contains only vertices of degree at least 3.*

**Proof.** If  $|A_1| = 1$  then by Lemma 4.1,  $(X_1 \setminus A_1, X_2 \cup A_1)$  is a connected non-path 2-join of  $G$ , contradicting the assumption that  $(X_1, X_2)$  is minimally sided connected non-path 2-join of  $G$ . So  $|A_1| \geq 2$ , and by symmetry  $|B_1| \geq 2$ . By Lemma 3.2(iii), all vertices of  $A_2 \cup B_2$  have degree at least 3.  $\square$

Recall that a *flat path* of a graph  $G$  is any path of  $G$  of length at least 2, whose interior vertices all have degree 2 in  $G$ , and whose ends have no common neighbors outside the path.

**Lemma 4.3.** Let  $G \in \mathcal{C}_{\text{No SC}}$  and let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a minimally sided connected non-path 2-join of  $G$ . Let  $X_1$  be a minimal side, and let  $P$  be a flat path of  $G$ . If  $P \cap X_1 \neq \emptyset$  and  $P \cap X_2 \neq \emptyset$ , then one of the following holds:

- (i) For an endvertex  $u$  of  $P$ ,  $P \setminus u \subseteq X_1$  and  $u \in A_2 \cup B_2$ .
- (ii) For endvertices  $u$  and  $v$  of  $P$ ,  $u \in A_2$ ,  $v \in B_2$ ,  $P \setminus \{u, v\} \subseteq X_1$ , the length of  $P$  is at least 3 and  $G[X_1]$  has exactly two connected components that are both a path with one end in  $A_1$ , one end in  $B_1$  and interior in  $C_1$ .

**Proof.** Let  $u$  and  $v$  be the endvertices of  $P$ . By Lemma 4.2, the interior of  $P$  must lie in  $X_1$  and w.l.o.g.  $u \in A_2$ . By Lemma 3.2(iii), the neighbor  $x$  of  $u$  along  $P$  has a neighbor in  $X_1$ , so  $|A_2| = 1$ . Hence if (i) does not hold then  $v \in B_2$ . Also,  $P$  is of length at least 3. So, the interior of  $P$  is a connected component  $C$  of  $G[X_1]$ . If  $G[X_1 \setminus C]$  is not a path with one end in  $A_1$ , one end in  $B_1$  and interior in  $C_1$  then  $(X_1 \setminus C, X_2 \cup C)$  is a connected non-path 2-join. Indeed, by Lemma 3.2(i),  $X_1 \setminus C$  meets  $A_1$  and  $B_1$ , and since it is not a path, it has size at least 3, so  $(X_1 \setminus C, X_2 \cup C)$  is a non-path 2-join that is connected by Lemma 3.2(ii). This contradicts  $(X_1, X_2)$  being minimally-sided. Therefore (ii) holds.  $\square$

**Lemma 4.4.** Let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a minimally-sided connected non-path 2-join of a graph  $G$ , with  $X_1$  being a minimal side. Assume that  $G$  and all the blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$  whose marker paths are of length at least 3, all belong to  $\mathcal{C}_{\text{No SC}}$ . Then  $(X_1, X_2)$  is an extreme 2-join and  $X_1$  is an extreme side.

**Proof.** Suppose that the block of decomposition  $G_1^{k_1}$ ,  $k_1 \geq 3$ , has a connected non-path 2-join with split  $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$ . For  $i = 1, 2$  let  $C'_i = X'_i \setminus (A'_i \cup B'_i)$ . Let  $P_2 = x_0 - x_1 - \dots - x_{k_1}$ , where  $x_0 = a_2$  and  $x_{k_1} = b_2$ , be the marker path of  $G_1^{k_1}$ .

**Case 1.** For some  $i \in \{1, 2\}$ ,  $P_2 \subseteq X'_i$ .

W.l.o.g.  $P_2 \subseteq X'_2$ . Note that since  $N_{G_1^{k_1}}(P_2 \setminus \{a_2, b_2\}) \subseteq \{a_2, b_2\}$  we have  $P_2 \cap (A'_2 \cup B'_2) \subseteq \{a_2, b_2\}$ . Note also that since  $a_2$  and  $b_2$  have no common neighbor in  $G_1^{k_1}$  we have  $|A'_2 \cap \{a_2, b_2\}| \leq 1$  and  $|B'_2 \cap \{a_2, b_2\}| \leq 1$ . So by symmetry it suffices to consider the following subcases:

**Case 1.1.**  $P_2 \subseteq C'_2$ .

Since  $a_2$  is adjacent to all vertices of  $A_1$  and  $a_2$  has no neighbor in  $X'_1$  (since  $a_2 \notin A'_2 \cup B'_2$ ) it follows that  $A_1 \subseteq X'_2$ . Similarly  $B_1 \subseteq X'_2$ . But then  $(X'_1, (X'_2 \setminus P_2) \cup X_2)$  is a connected non-path 2-join of  $G$ . Since  $A_1 \cup B_1 \subseteq X'_2$ ,  $X'_1 \subsetneq X_1$ , contradicting our choice of  $(X_1, X_2)$ .

**Case 1.2.**  $a_2 \in A'_2$  and  $P_2 \setminus \{a_2\} \subseteq C'_2$ .

So  $b_2$  has no neighbor in  $X'_1$ , and since  $b_2$  is adjacent to all vertices of  $B_1$ , it follows that  $B_1 \subseteq X'_2$ . In particular,  $X'_1 \subsetneq X_1$ . Since  $a_2 \in A'_2$ ,  $P_2 \subseteq X'_2$  and  $N_{G_1^{k_1}}(a_2) \setminus P_2 = A_1$ , it follows that  $A'_1 \subseteq A_1$  and  $(X'_1 \setminus A'_1) \cap A_1 = \emptyset$ . But then  $(X'_1, (X'_2 \setminus P_2) \cup X_2, A'_1, B'_1, (A'_2 \setminus \{a_2\}) \cup A_2, B'_2)$  is a split of a connected non-path 2-join of  $G$ , contradicting our choice of  $(X_1, X_2)$ .

**Case 1.3.**  $a_2 \in A'_2$ ,  $b_2 \in B'_2$  and  $P_2 \setminus \{a_2, b_2\} \subseteq C'_2$ .

Since  $(X'_1, X'_2)$  is not a path 2-join,  $X'_2 \cap X_1 \neq \emptyset$ , and hence  $X'_1 \subsetneq X_1$ . Since  $a_2 \in A'_2$ ,  $P_2 \subseteq X'_2$  and  $N_{G_1^{k_1}}(a_2) \setminus P_2 = A_1$ , it follows that  $A'_1 \subseteq A_1$  and  $(X'_1 \setminus A'_1) \cap A_1 = \emptyset$ . Similarly  $B'_1 \subseteq B_1$  and  $(X'_1 \setminus B'_1) \cap B_1 = \emptyset$ . But then  $(X'_1, (X'_2 \setminus P_2) \cup X_2, A'_1, B'_1, (A'_2 \setminus \{a_2\}) \cup A_2, (B'_2 \setminus \{b_2\}) \cup B_2)$  is a split of a connected non-path 2-join of  $G$ , contradicting our choice of  $(X_1, X_2)$ .

**Case 2.** For  $i = 1, 2$ ,  $P_2 \cap X'_i \neq \emptyset$ .

We may assume w.l.o.g. that  $(X'_1, X'_2)$  is a minimally-sided connected non-path 2-join of  $G_1^{k_1}$ , with  $X'_1$  being the minimal side. But then by Lemma 4.3 applied to  $(X'_1, X'_2)$  and  $P_2$  it suffices up to symmetry to consider the following two cases:

**Case 2.1.**  $a_2 \in A'_2$  and  $P_2 \setminus a_2 \subseteq X'_1$ .

Since  $x_1$  is of degree 2 and  $x_2 \in X'_1$ , it follows that  $|A'_2| = 1$ . Note that  $P_2 \setminus \{a_2, x_1\} \subseteq X'_1 \setminus A'_1$ . Since  $a_2 \in A'_2$ ,  $A'_1 \setminus \{x_1\} \subseteq A_1$  and  $A_1 \setminus A'_1 \subseteq X'_2 \setminus A'_2$ . Note that since  $(X'_1, X'_2)$  is connected,  $A_1 \setminus A'_1 \neq \emptyset$ . Note that  $P_2 \cap B'_1 \subseteq \{b_2\}$ .

First suppose that  $b_2 \in C'_1$ . Since  $b_2$  is adjacent to all vertices of  $B_1$ ,  $B_1 \subseteq X'_1$ . By Lemma 3.2(v), no vertex of  $B'_2$  has a neighbor in  $A'_2$ , which implies  $B'_2 \cap A_1 = \emptyset$ . So,  $C'_2 \neq \emptyset$  and by Lemma 3.2(vi),  $|X'_2| \geq 4$ . But then  $(X'_2 \setminus \{a_2\}, (X'_1 \setminus P_2) \cup X_2, A_1 \setminus A'_1, B'_2, A_2, B'_1)$  is a split of a connected non-path 2-join of  $G$ , contradicting our choice of  $(X_1, X_2)$  (since clearly  $X'_2 \setminus \{a_2\} \subsetneq X_1$ ).

Hence  $b_2 \notin C'_1$  and  $b_2 \in B'_1$ . Then  $B'_2 \subseteq B_1$  and  $B_1 \setminus B'_2 \subseteq X'_1$ . By Lemma 3.2(v), no vertex of  $B'_2$  has a neighbor in  $A'_2$ , which implies  $B'_2 \cap A_1 = \emptyset$ . So  $(X'_2 \setminus \{a_2\}, (X'_1 \setminus P_2) \cup X_2, A_1 \setminus A'_1, B'_2, A_2, (B'_1 \setminus \{b_2\}) \cup B_2)$  is a split of a connected non-path 2-join of  $G$ , contradicting our choice of  $(X_1, X_2)$  (since clearly  $X'_2 \setminus \{a_2\} \subsetneq X_1$ ).

**Case 2.2.**  $a_2 \in A'_2$ ,  $b_2 \in B'_2$  and  $P_2 \setminus \{a_2, b_2\} \subseteq X'_1$ .

Then  $x_1 \in A'_1$ ,  $x_{k_1-1} \in B'_1$  and  $P \setminus \{a_2, b_2, x_1, x_{k_1-1}\} \subseteq C'_1$ . Since  $x_1$  and  $x_{k_1-1}$  are of degree 2 in  $G_1^{k_1}$ , it follows that  $A'_2 = \{a_2\}$  and  $B'_2 = \{b_2\}$ . Since  $N_{G_1^{k_1}}(a_2) = A_1 \cup \{x_1\}$ , it follows that  $A'_1 \setminus \{x_1\} \subseteq A_1$  and  $A_1 \setminus A'_1 \subseteq X'_2$ . Similarly  $B'_1 \setminus \{x_{k_1-1}\} \subseteq B_1$  and  $B_1 \setminus B'_1 \subseteq X'_2$ . Since  $(X'_1, X'_2)$  is connected,  $A_1 \setminus A'_1 \neq \emptyset$  and  $B_1 \setminus B'_1 \neq \emptyset$ . Since  $a_2$  and  $b_2$  have no common neighbor in  $G_1^{k_1}$ ,  $(A_1 \setminus A'_1) \cap (B_1 \setminus B'_1) = \emptyset$ . Since  $(X'_1, X'_2)$  is a non-path 2-join,  $|X'_2 \setminus \{a_2, b_2\}| \geq 3$ . But then  $((X'_1 \setminus P_2) \cup X_2, X'_2 \setminus \{a_2, b_2\}, A_2, B_2, A_1 \setminus A'_1, B_1 \setminus B'_1)$  is a split of a 2-join of  $G$ . By Lemma 3.2(ii) this 2-join is connected. Since  $G_1^{k_1}[X'_2]$  and  $G[X_2]$  are not paths, this 2-join is a non-path 2-join. But then since  $X'_2 \setminus \{a_2, b_2\} \subsetneq X_1$ , our choice of  $(X_1, X_2)$  is contradicted.  $\square$

When  $\mathcal{M}$  is a collection of vertex-disjoint flat paths, a 2-join  $(X_1, X_2)$  is  $\mathcal{M}$ -independent if for every path  $P$  from  $\mathcal{M}$  we have either  $V(P) \subseteq X_1$  or  $V(P) \subseteq X_2$ . These special types of 2-joins will have a fundamental role to play when it comes to computing a largest stable set.

**Lemma 4.5.** Let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a minimally-sided connected non-path 2-join of a graph  $G$ , with  $X_1$  being a minimal side. Assume that  $G$  and all the blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$  whose marker paths are of length at least 3, all belong to  $\mathcal{C}_{\text{no sc}}$ . Let  $\mathcal{M}$  be a set of vertex-disjoint flat paths of length at least 3 of  $G$ . If there exists a path  $P \in \mathcal{M}$  such that  $P \cap A_1 \neq \emptyset$  and  $P \cap A_2 \neq \emptyset$ , then let  $A'_1 = A_2$ , and otherwise let  $A'_1 = A_1$ . If there exists a path  $P \in \mathcal{M}$  such that  $P \cap B_1 \neq \emptyset$  and  $P \cap B_2 \neq \emptyset$ , then let  $B'_1 = B_2$ , and otherwise let  $B'_1 = B_1$ . Let  $X'_1 = X_1 \cup A'_1 \cup B'_1$  and  $X'_2 = V(G) \setminus X'_1$ . Then the following hold:

- (i)  $(X'_1, X'_2)$  is a connected non-path 2-join of  $G$ .
- (ii)  $(X'_1, X'_2)$  is  $\mathcal{M}$ -independent.
- (iii)  $(X'_1, X'_2)$  is an extreme 2-join of  $G$  and  $X'_1$  is an extreme side of this 2-join.

**Proof.** If there exists a path  $P \in \mathcal{M}$  such that  $P \cap A_1 \neq \emptyset$  and  $P \cap A_2 \neq \emptyset$ , then by Lemma 4.3, either for an endvertex  $u$  of  $P$ ,  $P \setminus u \subseteq X_1$  and  $u \in A_2$ ; or for endvertices  $u$  and  $v$  of  $P$ ,  $u \in A_2$ ,  $v \in B_2$  and  $P \setminus \{u, v\} \subseteq X_1$ . Since the intermediate vertices of  $P$  are of degree 2, it follows that  $|A_2| = 1$ , and so (i) holds by Lemma 4.1 (possibly applied twice).

Let  $(X'_1, X'_2, A'_1, B'_1, A'_2, B'_2)$  be the split of  $(X'_1, X'_2)$ , where  $A_2 \in \{A'_1, A'_2\}$  and  $B_2 \in \{B'_1, B'_2\}$ . By Lemma 4.3, applied to  $(X'_1, X'_2)$ , paths  $P \in \mathcal{M}$  are one of the following types:

Type 1:  $P \subseteq X_1$ .

Type 2:  $P \subseteq X_2$ .

Type 3: For an endvertex  $u$  of  $P$ ,  $u \in A_2$  and  $P \setminus u \subseteq X_1$ .

Type 4: For an endvertex  $u$  of  $P$ ,  $u \in B_2$  and  $P \setminus u \subseteq X_1$ .

Type 5: For endvertices  $u$  and  $v$  of  $P$ ,  $u \in A_2$ ,  $v \in B_2$  and  $P \setminus \{u, v\} \subseteq X_1$ .

Note that since  $\mathcal{M}$  is a collection of vertex-disjoint paths, at most one path of  $\mathcal{M}$  is of type 3 (resp. type 4), and if there exists a type 3 (resp. type 4) path then for every type 2 path  $P$ ,  $P \cap A_2 = \emptyset$  (resp.  $P \cap B_2 = \emptyset$ ). Also there is at most one type 5 path in  $\mathcal{M}$ , and if such a path exists there are no type 3 and 4 paths in  $\mathcal{M}$ , and for every type 2 path  $P$  of  $\mathcal{M}$ ,  $P \cap A_2 = \emptyset$  and  $P \cap B_2 = \emptyset$ . So by the construction of  $(X'_1, X'_2)$  all type 1 (resp. type 2) paths w.r.t.  $(X_1, X_2)$  stay type 1 (resp. type 2) w.r.t.  $(X'_1, X'_2)$ , and all type 3, 4 and 5 paths w.r.t.  $(X_1, X_2)$  become type 1 w.r.t.  $(X'_1, X'_2)$ . Therefore (ii) holds.

For  $k_1, k_2 \geq 3$ , let  $G_1^{k_1}$  and  $G_2^{k_2}$  be the blocks of decomposition of  $G$  w.r.t. the 2-join  $(X_1, X_2)$ . By Lemma 4.4,  $G_1^{k_1}$  has no connected non-path 2-join. Let  $P_2$  be the marker path of  $G_1^{k_1}$ . Let  $G_1'^{k_1}$  be the block of decomposition of  $G$  with respect to  $(X'_1, X'_2)$ . Notice that  $G_1'^{k_1}$  can be obtained from  $G_1^{k_1}$  by subdividing an edge of  $P_2$  (0, 1 or 2 times), and hence by our assumption,  $G_1'^{k_1} \in \mathcal{C}_{\text{No SC}}$ . Therefore by Lemma 4.4,  $G_1'^{k_1}$  has no connected non-path 2-join and hence (iii) holds.  $\square$

**Lemma 4.6.** *There is an algorithm with the following specification:*

**Input:** A connected graph  $G$  and a set  $\mathcal{M}$  of vertex-disjoint flat paths of  $G$  of length at least 3.

**Output:** One of the following is returned.

- (i) An extreme connected non-path 2-join of  $G$  (with an identified extreme side) that is  $\mathcal{M}$ -independent.
- (ii)  $G$  or a block of decomposition of  $G$  w.r.t. some 2-join whose marker path is of length at least 3, has a star cutset.
- (iii)  $G$  has no connected non-path 2-join.

**Running time:**  $\mathcal{O}(n^3m)$ .

**Proof.** First check whether  $G$  has a star cutset. Note that this can be done in time  $\mathcal{O}(n^3)$  as noted by Chvátal [7]: for every  $x \in V(G)$ , check whether  $G \setminus N[x]$  is disconnected, and also check whether there exists  $y \in N(x)$  such that  $y$  has no neighbor in  $G \setminus N[x]$ . If the answer to any of these is yes, then  $G$  has a star cutset centered at  $x$ , and otherwise it does not. If  $G$  is identified as having a star cutset return (ii) and stop, and otherwise continue.

Note that at this point in the algorithm we know that  $G \in \mathcal{C}_{\text{No SC}}$ , and hence by Lemma 3.2(ii) any 2-join of  $G$  is connected.

Run the  $\mathcal{O}(n^3m)$ -algorithm from Theorem 5.2 in [2] for  $G$ . This algorithm outputs a minimally sided non-path 2-join of an input graph with no star cutset, or certifies that the input graph has no non-path 2-join. If this stage of the algorithm does not find any non-path 2-join in  $G$ , then return (iii) and stop. Otherwise let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be the split of a minimally-sided connected non-path 2-join found, and w.l.o.g. assume that  $X_1$  is a minimal side.

Let  $G_1^3$  and  $G_2^3$  be the blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$ . Check whether  $G_1^3$  and  $G_2^3$  have a star cutset. If any one of them does, then return (ii) and stop. Otherwise, we continue and we note that since  $G_1^3, G_2^3 \in \mathcal{C}_{\text{No SC}}$  clearly blocks of decomposition of  $G$  w.r.t.  $(X_1, X_2)$   $G_1^{k_1}$  and  $G_2^{k_2}$ , for any  $k_1, k_2 \geq 3$ , also belong to  $\mathcal{C}_{\text{No SC}}$ .

If there exists a path  $P \in \mathcal{M}$  such that  $P \cap A_1 \neq \emptyset$  and  $P \cap A_2 \neq \emptyset$ , then let  $A'_1 = A_2$ , and otherwise let  $A'_1 = A_1$ . If there exists a path  $P \in \mathcal{M}$  such that  $P \cap B_1 \neq \emptyset$  and  $P \cap B_2 \neq \emptyset$ , then let  $B'_1 = B_2$ , and otherwise let  $B'_1 = B_1$ . Let  $X'_1 = X_1 \cup A'_1 \cup B'_1$  and  $X'_2 = V(G) \setminus X'_1$ . Then by Lemma 4.5,  $(X'_1, X'_2)$  is an extreme connected non-path 2-join (with  $X'_1$  being an extreme side) that is  $\mathcal{M}$ -independent. We return this 2-join in (i) and stop.

Clearly this algorithm can be implemented to run in time  $\mathcal{O}(n^3m)$ .  $\square$



Note that later when we apply the above algorithm in our main algorithm, if the output is (ii), then by Lemma 3.7 we can conclude that  $G$  does not belong to  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$  or to  $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$ .

## 5. Keeping track of cliques

Here we show how to find a maximum clique in a graph using 2-joins. For the sake of induction we have to solve the weighted version of the problem.

Through all the next sections, by graph we mean a graph with weights on the vertices. Weights are numbers from  $K$  where  $K$  means either the set  $\mathbb{R}_+$  of non-negative real numbers or the set  $\mathbb{N}_+$  of non-negative integers. The statements of the theorems will be true for  $K = \mathbb{R}_+$  but the algorithms are to be implemented with  $K = \mathbb{N}_+$ . Let  $G$  be a weighted graph with a weight function  $w$  on  $V(G)$ . When  $H$  is an induced subgraph of  $G$  or a subset of  $V(G)$ ,  $w(H)$  denotes the sum of the weights of vertices in  $H$ . Note that we view a graph where no weight is assigned to the vertices as a weighted graph whose vertices have all weight 1. Here,  $\omega(G)$  denotes the weight of a maximum weighted clique of  $G$ .

Let  $(X_1, X_2, A_1, B_1, A_2, B_2)$  be a split of a connected 2-join of  $G$ . We define for  $k \geq 3$  the *clique-block*  $G_2^k$  of  $G$  with respect to  $(X_1, X_2)$ . It is obtained from the block  $G_2^k$  by giving weights to the vertices. Let  $P_1 = a_1 - x_1 - \dots - x_{k-1} - b_1$  be the marker path of  $G_2^k$ . We assign the following weights to the vertices of  $G_2^k$ :

- for every  $u \in X_2$ ,  $w_{G_2^k}(u) = w_G(u)$ ;
- $w_{G_2^k}(a_1) = \omega(G[A_1])$ ;
- $w_{G_2^k}(b_1) = \omega(G[B_1])$ ;
- $w_{G_2^k}(x_1) = \omega(G[X_1]) - \omega(G[A_1])$ ;
- $w_{G_2^k}(x_i) = 0$ , for  $i = 2, \dots, k-1$ .

**Lemma 5.1.**  $\omega(G) = \omega(G_2^k)$ .

**Proof.** Let  $K$  be a maximum weighted clique of  $G$ . We show that the clique-block  $G_2^k$  has a clique of weight  $w_G(K)$ , and hence  $\omega(G) \leq \omega(G_2^k)$ . If  $K \subseteq X_2$  then  $K \subseteq V(G_2^k)$ . If  $K \subseteq X_1$  then  $\{a_1, x_1\}$  is a clique of  $G_2^k$  of weight  $w_G(K)$ . So assume that  $K \cap X_1 \neq \emptyset$  and  $K \cap X_2 \neq \emptyset$ . W.l.o.g.  $K \cap A_1 \neq \emptyset$  and  $K \cap A_2 \neq \emptyset$ , and hence  $K \subseteq A_1 \cup A_2$ . But then  $(K \setminus A_1) \cup \{a_1\}$  is a clique of  $G_2^k$  of weight  $w_G(K)$ . Therefore  $\omega(G) \leq \omega(G_2^k)$ .

Now let  $K$  be a maximum weighted clique of  $G_2^k$ . We show that  $G$  has a clique of weight  $w_{G_2^k}(K)$ , and hence  $\omega(G_2^k) \leq \omega(G)$ . If  $K \subseteq X_2$  then  $K$  is a clique of  $G$ . Suppose  $K \cap P_1 = \{a_1\}$ , and let  $K'$  be a clique of  $A_1$  whose weight is  $\omega(G[A_1])$ . Then  $(K \cap A_2) \cup K'$  is a clique of  $G$  of weight  $w_{G_2^k}(K)$ . So we may assume that  $K = \{a_1, x_1\}$ . Then  $w_{G_2^k}(K) = \omega(G[X_1])$ , and  $G$  has a clique of the same weight. Therefore  $\omega(G_2^k) \leq \omega(G)$ .  $\square$

## 6. Keeping track of stable sets

Here we show how to use 2-joins to compute maximum stable sets. This is more difficult than cliques mainly because stable sets may completely overlap both sides of a 2-join. For the sake of induction we need to put weights on the vertices. But even with weights, there is an issue: we are not able to compute maximum weighted stable set of a graph assuming that some computations are done on its blocks as defined in Section 3. So we need to enlarge slightly our blocks to encode information, and this causes some trouble. First, the extended blocks may fail to be in the class we are working on. This problem will be solved in Section 8 by building the decomposition tree in two steps. Also in a decomposition tree built with our unusual blocks, the leaves may fail to be basic graphs, so computing something in the leaves of the tree is a problem postponed to Section 7.

Throughout this section,  $G$  is a fixed graph with a weight function  $w$  on the vertices and  $(X_1, X_2, A_1, B_1, A_2, B_2)$  is a split of a 2-join of  $G$ . For  $i = 1, 2$ ,  $C_i = X_i \setminus (A_i \cup B_i)$ . For any graph  $H$ ,  $\alpha(H)$  denotes the weight of a maximum weighted stable set of  $H$ . We define  $a = \alpha(G[A_1 \cup C_1])$ ,  $b = \alpha(G[B_1 \cup C_1])$ ,  $c = \alpha(G[C_1])$  and  $d = \alpha(G[X_1])$ .

**Lemma 6.1.** *Let  $S$  be a stable set of  $G$  of maximum weight. Then one and only one of the following holds:*

- (i)  $S \cap A_1 \neq \emptyset$ ,  $S \cap B_1 = \emptyset$ ,  $S \cap X_1$  is a maximum weighted stable set of  $G[A_1 \cup C_1]$  and  $w(S \cap X_1) = a$ ;
- (ii)  $S \cap A_1 = \emptyset$ ,  $S \cap B_1 \neq \emptyset$ ,  $S \cap X_1$  is a maximum weighted stable set of  $G[B_1 \cup C_1]$  and  $w(S \cap X_1) = b$ ;
- (iii)  $S \cap A_1 = \emptyset$ ,  $S \cap B_1 = \emptyset$ ,  $S \cap X_1$  is a maximum weighted stable set of  $G[C_1]$  and  $w(S \cap X_1) = c$ ;
- (iv)  $S \cap A_1 \neq \emptyset$ ,  $S \cap B_1 \neq \emptyset$ ,  $S \cap X_1$  is a maximum weighted stable set of  $G[X_1]$  and  $w(S \cap X_1) = d$ .

**Proof.** Follows directly from the definition of a 2-join.  $\square$

### 6.1. Stable sets overlapping 2-joins

We need kinds of blocks that preserve being in  $C^{\text{BERGE}}$ . To define them we need several inequalities that tell more about how stable sets and 2-joins overlap.

**Lemma 6.2.**  $0 \leq c \leq a$ ,  $b \leq d \leq a + b$ .

**Proof.** The inequalities  $0 \leq c \leq a$ ,  $b \leq d$  are trivially true. Let  $D$  be a maximum weighted stable set of  $G[X_1]$ . We have:

$$d = w(D) = w(D \cap A_1) + w(D \cap (C_1 \cup B_1)) \leq a + b. \quad \square$$

A 2-join with split  $(X_1, X_2, A_1, B_1, A_2, B_2)$  is said to be  $X_1$ -even (resp.  $X_1$ -odd) if all paths from  $A_1$  to  $B_1$  with interior in  $C_1$  are of even length (resp. odd length). Note that from Lemma 3.1, if  $G$  is in  $C^{\text{PARITY}}$  and  $(X_1, X_2)$  is connected, then  $(X_1, X_2)$  must be either  $X_1$ -even or  $X_1$ -odd.

**Lemma 6.3.** *If  $(X_1, X_2)$  is an  $X_1$ -even 2-join of  $G$ , then  $a + b \leq c + d$ .*

**Proof.** Let  $A$  be a stable set of  $G[A_1 \cup C_1]$  of weight  $a$  and  $B$  a stable set of  $G[B_1 \cup C_1]$  of weight  $b$ . In the bipartite graph  $G[A \cup B]$ , we denote by  $Y_A$  (resp.  $Y_B$ ) the set of those vertices of  $A \cup B$  such that there exists a path in  $G[A \cup B]$  joining them to some vertex of  $A \cap A_1$  (resp.  $B \cap B_1$ ). Note that from the definition,  $A \cap A_1 \subseteq Y_A$ ,  $B \cap B_1 \subseteq Y_B$  and no edges exist between  $Y_A \cup Y_B$  and  $(A \cup B) \setminus (Y_A \cup Y_B)$ . Also,  $Y_A$  and  $Y_B$  are disjoint with no edges between them because else, there is some path in  $G[A \cup B]$  from some vertex of  $A \cap A_1$  to some vertex of  $B \cap B_1$ . If such a path is minimal with respect to this property, its interior is in  $C_1$  and it is of odd length because  $G[A \cup B]$  is bipartite. This contradicts the assumption that  $(X_1, X_2)$  is  $X_1$ -even. Now we put:

- $Z_D = (A \cap Y_A) \cup (B \cap Y_B) \cup (A \setminus (Y_A \cup Y_B))$ ;
- $Z_C = (A \cap Y_B) \cup (B \cap Y_A) \cup (B \setminus (Y_A \cup Y_B))$ .

From all the definitions and properties above,  $Z_D$  and  $Z_C$  are stable sets and  $Z_D \subseteq X_1$  and  $Z_C \subseteq C_1$ . So,  $a + b = w(Z_C) + w(Z_D) \leq c + d$ .  $\square$

**Lemma 6.4.** *If  $(X_1, X_2)$  is an  $X_1$ -odd 2-join of  $G$ , then  $c + d \leq a + b$ .*

**Proof.** Let  $D$  be a stable set of  $G[X_1]$  of weight  $d$  and  $C$  a stable set of  $G[C_1]$  of weight  $c$ . In the bipartite graph  $G[C \cup D]$ , we denote by  $Y_A$  (resp.  $Y_B$ ) the set of those vertices of  $C \cup D$  such that there exists a path in  $G[C \cup D]$  joining them to some vertex of  $D \cap A_1$  (resp.  $D \cap B_1$ ). Note that from the definition,  $D \cap A_1 \subseteq Y_A$ ,  $D \cap B_1 \subseteq Y_B$  and no edges exist between  $Y_A \cup Y_B$  and  $(C \cup D) \setminus (Y_A \cup Y_B)$ .

Also,  $Y_A$  and  $Y_B$  are disjoint with no edges between them because else, there is some path in  $G[C \cup D]$  from some vertex of  $D \cap A_1$  to some vertex of  $D \cap B_1$ . If such a path is minimal with respect to this property, its interior is in  $C_1$  and it is of even length because  $G[C \cup D]$  is bipartite. This contradicts the assumption that  $(X_1, X_2)$  is  $X_1$ -odd. Now we put:

- $Z_A = (D \cap Y_A) \cup (C \cap Y_B) \cup (C \setminus (Y_A \cup Y_B))$ ;
- $Z_B = (D \cap Y_B) \cup (C \cap Y_A) \cup (D \setminus (Y_A \cup Y_B))$ .

From all the definitions and properties above,  $Z_A$  and  $Z_B$  are stable sets and  $Z_A \subseteq A_1 \cup C_1$  and  $Z_B \subseteq B_1 \cup C_1$ . So,  $c + d = w(Z_A) + w(Z_B) \leq a + b$ .  $\square$

## 6.2. Even and odd blocks

We call *flat claw* of a weighted graph  $G$  any set  $\{q_1, q_2, q_3, q_4\}$  of vertices such that:

- the only edges between the  $q_i$ 's are  $q_1q_2$ ,  $q_2q_3$  and  $q_4q_2$ ;
- $q_1$  and  $q_3$  have no common neighbor in  $V(G) \setminus \{q_2\}$ ;
- $q_4$  has degree 1 in  $G$  and  $q_2$  has degree 3 in  $G$ .

**Lemma 6.5.** *Let  $G$  be a graph,  $Q = \{q_1, q_2, q_3, q_4\}$  a flat claw of  $G$  and  $S'$  a maximum weighted stable set of  $G$ . Then one and only one of the following holds:*

- (i)  $q_1 \in S'$ ,  $q_3 \notin S'$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{q_1, q_2, q_4\}]$ ;
- (ii)  $q_1 \notin S'$ ,  $q_3 \in S'$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{q_2, q_3, q_4\}]$ ;
- (iii)  $q_1 \notin S'$ ,  $q_3 \notin S'$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{q_2, q_4\}]$ ;
- (iv)  $q_1 \in S'$ ,  $q_3 \in S'$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{q_1, q_2, q_3, q_4\}]$ .

**Proof.** Follows directly from the definitions.  $\square$

We define now the *even block*  $G_2$  with respect to  $(X_1, X_2)$ . We keep  $X_2$  and replace  $X_1$  by a flat claw on  $q_1, \dots, q_4$  where  $q_1$  is complete to  $A_2$  and  $q_3$  is complete to  $B_2$ . We give the following weights:  $w(q_1) = d - b$ ,  $w(q_2) = c$ ,  $w(q_3) = d - a$ ,  $w(q_4) = a + b - d$ . From Lemma 6.2, all weights are non-negative. By Lemma 6.3, the following lemma applies in particular if  $(X_1, X_2)$  is a connected  $X_1$ -even 2-join.

**Lemma 6.6.** *If  $a + b \leq c + d$  and if  $G_2$  is the even block of  $G$ , then  $\alpha(G_2) = \alpha(G)$ .*

**Proof.** Let  $S$  be a stable set of maximum weight in  $G$ . Then  $S$  must satisfy one of (i), (ii), (iii) or (iv) of Lemma 6.1. Respective to these cases one can construct a stable set  $S'$  of  $G_2$  that has the weight of  $S$ , by taking the union of  $S \cap X_2$  and one of  $\{q_1, q_4\}$ ,  $\{q_3, q_4\}$ ,  $\{q_2\}$  or  $\{q_1, q_3, q_4\}$ .

Conversely, if  $S'$  is a stable set of  $G_2$  of maximum weight then it satisfies one of (i), (ii), (iii) or (iv) of Lemma 6.5. Respective to these cases,  $w(S' \cap Q)$  is  $a$ ,  $b$ ,  $c$  or  $d$  (by Lemma 6.2 and because  $a + b \leq c + d$ ) and one can construct a maximum stable set  $S$  of  $G$  by replacing  $S' \cap Q$  by a maximum weighted stable set of  $G[A_1 \cup C_1]$ ,  $G[B_1 \cup C_1]$ ,  $G[C_1]$  or  $G[X_1]$ .  $\square$

We call *flat vault* of graph  $G$  any set  $\{r_1, r_2, r_3, r_4, r_5, r_6\}$  of vertices such that:

- the only edges between the  $r_i$ 's are such that  $r_3, r_4, r_5, r_6, r_3$  is a 4-hole;
- $N(r_1) = N(r_5) \setminus \{r_4, r_6\}$ ;
- $N(r_2) = N(r_6) \setminus \{r_3, r_5\}$ ;
- $r_1$  and  $r_2$  have no common neighbors;
- $r_3$  and  $r_4$  have degree 2 in  $G$ .

**Lemma 6.7.** Let  $G$  be a graph,  $Q = \{r_1, r_2, r_3, r_4, r_5, r_6\}$  a flat vault of  $G$  and  $S'$  a maximum weighted stable set of  $G$ . Then one and only one of the following holds:

- (i)  $S' \cap \{r_1, r_5\} \neq \emptyset$ ,  $S' \cap \{r_2, r_6\} = \emptyset$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{r_1, r_3, r_4, r_5\}]$ ;
- (ii)  $S' \cap \{r_1, r_5\} = \emptyset$ ,  $S' \cap \{r_2, r_6\} \neq \emptyset$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{r_2, r_3, r_4, r_6\}]$ ;
- (iii)  $S' \cap \{r_1, r_5\} = \emptyset$ ,  $S' \cap \{r_2, r_6\} = \emptyset$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{r_3, r_4\}]$ ;
- (iv)  $S' \cap \{r_1, r_5\} \neq \emptyset$ ,  $S' \cap \{r_2, r_6\} \neq \emptyset$  and  $S' \cap Q$  is a maximum weighted stable set of  $G[\{r_1, r_2, r_3, r_4, r_5, r_6\}]$ .

**Proof.** Follows directly from the definitions.  $\square$

Let us now define the odd block  $G_2$  with respect to  $(X_1, X_2)$ . We replace  $X_1$  by a flat vault on  $r_1, \dots, r_6$ . Moreover  $r_1, r_5$  are complete to  $A_2$  and  $r_2, r_6$  are complete to  $B_2$ . We give the following weights:  $w(r_1) = d - b$ ,  $w(r_2) = d - a$ ,  $w(r_3) = w(r_4) = c$ ,  $w(r_5) = w(r_6) = a + b - c - d$ . Note that if we suppose  $c + d \leq a + b$  (which holds in particular if  $(X_1, X_2)$  is an  $X_1$ -odd connected 2-join by Lemma 6.4), all the weights are non-negative by Lemma 6.2.

**Lemma 6.8.** If  $c + d \leq a + b$  and if  $G_2$  is the odd block of  $G$ , then  $\alpha(G_2) = \alpha(G)$ .

**Proof.** Let  $S$  be a stable set of maximum weight in  $G$ . Then  $S$  must satisfy one of (i), (ii), (iii) or (iv) of Lemma 6.1. So, respective to these cases, it is easy to construct a stable set  $S'$  of  $G_2$  that has the weight of  $S$ , by taking the union of  $S \cap X_2$  and one of  $\{r_1, r_3, r_5\}$ ,  $\{r_2, r_4, r_6\}$ ,  $\{r_3\}$  or  $\{r_1, r_2, r_3, r_5\}$ .

Conversely, if  $S'$  is a stable set of  $G_2$  of maximum weight then it satisfies one of (i), (ii), (iii) or (iv) of Lemma 6.7. Respective to these cases,  $w(S' \cap Q)$  is  $a$ ,  $b$ ,  $c$  or  $d$  (because  $c + d \leq a + b$ ) and one can construct a maximum weighted stable set  $S$  of  $G$  of the same weight as  $S'$  by replacing  $S' \cap \{r_1, r_2, r_3, r_4, r_5, r_6\}$  by a maximum weighted stable set of  $G[A_1 \cup C_1]$ ,  $G[B_1 \cup C_1]$ ,  $G[C_1]$  or  $G[X_1]$ .  $\square$

Note that the following lemma fails for  $\mathcal{C}^{\text{EHF}}$  because the odd block contains an even hole.

**Lemma 6.9.** Let  $G$  be a graph in  $\mathcal{C}^{\text{BERGE}}$  and  $(X_1, X_2)$  be a connected 2-join of  $G$ . If  $(X_1, X_2)$  is  $X_1$ -even then the even block  $G_2$  is in  $\mathcal{C}^{\text{BERGE}}$ . If  $(X_1, X_2)$  is  $X_1$ -odd then the odd block  $G_2$  is in  $\mathcal{C}^{\text{BERGE}}$ .

**Proof.** Suppose that  $G_2$  contains an odd hole  $H$ . If no edge of  $H$  has both ends in  $V(G_2) \setminus X_2$ , then  $H \subseteq X_2 \cup (N_{G_2}(A_2) \setminus X_2) \cup (N_{G_2}(B_2) \setminus X_2)$ . We obtain an odd hole  $H'$  of  $G$  as follows. By Lemma 3.2(iv), there exist non-adjacent vertices  $a_1 \in A_1$ ,  $b_1 \in B_1$ . If  $H \cap (N_{G_2}(A_2) \setminus X_2) \neq \emptyset$ , we replace the unique vertex in  $H \cap (N_{G_2}(A_2) \setminus X_2)$  by  $a_1$ . We proceed similarly with  $H \cap (N_{G_2}(B_2) \setminus X_2)$  and  $b_1$ . We obtain an odd hole  $H'$  of  $G$ , a contradiction.

If  $H$  has an edge whose ends are both in  $V(G_2) \setminus X_2$  then  $H$  is vertex-wise partitioned into  $q_1 - q_2 - q_3$  when  $(X_1, X_2)$  is  $X_1$ -even (resp.  $r_5 - r_6$  when  $(X_1, X_2)$  is  $X_1$ -odd), and a path with one end in  $A_2$ , one end in  $B_2$  and interior in  $C_2$ . Then an odd hole of  $G$  can be obtained by replacing  $q_1 - q_2 - q_3$  (resp.  $r_5 - r_6$ ) by a path of even (resp. odd) length of  $G$  from  $A_1$  to  $B_1$  with interior in  $C_1$ . This contradicts  $G$  being Berge.

Suppose that  $G_2$  contains an odd antihole  $H$ . Since an antihole on 5 vertices is in fact a hole, we may assume that  $H$  is on at least 7 vertices. So all vertices of  $H$  have degree at least four. Hence, if  $G_2$  is an even block then  $H$  cannot go through  $q_2, q_4$ . So, up to the replacement of at most two vertices,  $H$  is an odd antihole of  $G$ , a contradiction. Now suppose  $G_2$  is an odd block. Because of the degrees,  $r_3, r_4 \notin H$ . In an antihole on at least 7 vertices, every pair of vertices has a common neighbor. A vertex of  $\{r_1, r_5\}$  has no common neighbor with a vertex of  $\{r_2, r_6\}$ . So, we may assume that  $H \cap \{r_2, r_6\} = \emptyset$ . We have  $N_{G_2}(r_1) \subseteq N_{G_2}(r_5)$  so not both  $r_1, r_5$  are in  $H$ . So, we may assume that  $r_5 \notin H$ . So, up to the replacement of  $r_1$  by a vertex of  $A_1$ ,  $H$  is an odd antihole of  $G$ , a contradiction.  $\square$

### 6.3. The gem block

We present here a block of decomposition that we do not use in the rest of the paper but that is interesting because it can be used in all situations (whereas some inequalities must be satisfied for even and odd blocks).

To build the *gem-block*  $G_2$  replace  $X_1$  by an induced path  $p-x-y-p'$  plus a vertex  $z$  complete to this path. Vertex  $p$  is complete to  $A_2$  and vertex  $p'$  is complete to  $B_2$ . We give weights:  $w(p) = a$ ,  $w(x) = a + b - d$ ,  $w(y) = d$ ,  $w(p') = 2d - a$ ,  $w(z) = c + d$ . Note that all weights are non-negative by Lemma 6.2. We omit the proof of the following lemma since we do not use it.

**Lemma 6.10.** *If  $G_2$  is the gem-block of  $G$  then  $\alpha(G_2) = \alpha(G) + d$ .*

The gem-block appears implicitly in the proof of the NP-completeness result in Section 10.

## 7. Extensions of basic classes

To build a decomposition tree that allows keeping track of maximum stable sets we use the even and odd blocks defined in Section 6. As a consequence, the leaves of our decomposition tree may fail to be basic, but are what we call *extensions* of basic graphs. Let us define this.

Let  $P = p_1 - \dots - p_k$ ,  $k \geq 4$ , be a flat path of a graph  $G$ . *Extending*  $P$  means:

- Either:
  - (i) replace the vertices of  $P$  by a flat claw on  $q_1, \dots, q_4$  where  $q_1$  is complete to  $N_G(p_1) \setminus \{p_2\}$  and  $q_3$  is complete to  $N_G(p_k) \setminus \{p_{k-1}\}$ ;
  - (ii) replace  $X_1$  by a flat vault on  $r_1, \dots, r_6$  where  $r_1, r_5$  are complete to  $N_G(p_1) \setminus \{p_2\}$  and  $r_2, r_6$  are complete to  $N_G(p_k) \setminus \{p_{k-1}\}$ .
- Mark the vertices of the flat claw (or vault) with the integer  $k$ .

An *extension* of a pair  $(G, \mathcal{M})$ , where  $G$  is a graph and  $\mathcal{M}$  is a set of vertex-disjoint flat paths of length at least 3 of  $G$ , is any weighted graph obtained by extending the flat paths of  $\mathcal{M}$  and giving any non-negative weights to all the vertices. Note that since  $\mathcal{M}$  is a set of *vertex-disjoint* paths, the extensions of the paths from  $\mathcal{M}$  can be done in any order and lead to the same graph. An *extension* of a graph  $G$  is any graph that is an extension of  $(G, \mathcal{M})$  for some  $\mathcal{M}$ .

We say that the extension of  $P$  is *parity-preserving* when  $P$  has even length and is replaced by a flat claw, or when  $P$  has odd length and is replaced by a flat vault. We define the *parity-preserving* extension of a pair  $(G, \mathcal{M})$  and of a graph  $G$  by requiring that all extensions of paths are parity-preserving.

### 7.1. Recognition of extensions basic graphs

We will describe algorithms for computing cliques and stable sets in graphs from our basic classes and their extensions. To apply these algorithms we need to detect in which basic class a graph is. For bipartite graphs, line graphs of bipartite graphs and their complements, this is a classical problem, see [27,33]. For double split graphs, this is very easy, see [36], Section 7. For path-cobipartite graphs it is very easy by picking a vertex of degree 2 if any, checking if it belongs to a flat path, if so taking a maximal such flat path, and checking if the flat path satisfies the definition of a path-cobipartite graph. A similar trick recognizes path-double split graphs. All these classes can be recognized in linear time.

The class  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$  can be recognized in time  $\mathcal{O}(n^2m)$  by checking for all pairs of vertices if their deletion gives the line graph of a tree. Checking that the graph is even-hole free is easy, since in the line graph of a tree, there exists a unique induced path joining any pair of vertices.

Recognition of extensions of basic graphs is easy thanks to the mark given to the new vertices arising from extensions. These marks allow to compute the original graph from its extension.

Also, when a graph is identified to be basic, the algorithms above certify that the graph is basic. For bipartite graphs, it gives a bipartition, for a line graph  $G$ , a root-graph  $R$  such that  $G = L(R)$ . For double split graphs, path cobipartite graphs, path-double split graph and graph from  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ , the sets like in their respective definitions are output.

We do not write a theorem about these algorithms, but in the description of the algorithms in the rest of the paper, when we consider an extended basic graph, it is implicit that the algorithm can know in time  $\mathcal{O}(n^2m)$  in which basic class the graph is. Since all our algorithms run in time at least  $\mathcal{O}(n^3m)$ , this does not affect the overall complexity.

## 7.2. Parity-preserving extensions of basic Berge graphs (except line graphs)

**Lemma 7.1.** *A parity-preserving extension of a bipartite graph is a bipartite graph.*

**Proof.** Suppose that a graph  $G$  is bipartite. So we color its vertices black and white. Suppose that a parity-preserving extension with respect to  $P = p_1 - \dots - p_k$  is performed. If the path has even length then up to symmetry  $p_1$  and  $p_k$  are black. Since the extension is parity-preserving,  $P$  is replaced by a flat claw on  $q_1, q_2, q_3, q_4$ . We give color black to  $q_1, q_3, q_4$  and color white to  $q_2$ . If the path has odd length then up to symmetry  $p_1$  is black and  $p_k$  is white. Since the extension is parity-preserving,  $P$  is replaced by a flat vault on  $r_1, r_2, r_3, r_4, r_5, r_6$ . We give color black to  $r_1, r_3, r_5$  and color white to  $r_2, r_4, r_6$ . This shows that the parity-preserving extension of  $P$  yields a bipartite graphs and the lemma follows by an induction on the number of extended paths.  $\square$

The following lemma shows that maximum weighted stable sets can be computed for all parity-preserving extensions of Berge basic classes, except line graphs of bipartite graphs.

**Lemma 7.2.** *There is an algorithm with the following specification:*

**Input:** *A weighted graph  $G$  that is a parity preserving extension of either a bipartite graph, the complement of a bipartite graph, the complement of a line graph of a bipartite graph, a path-cobipartite graph, the complement of a path-cobipartite graph, a path-double split graph or the complement of a path-double split graph.*

**Output:** *A maximum weighted stable set of  $G$ .*

**Running time:**  $\mathcal{O}(n^5)$ .

**Proof.** For parity-preserving extensions of bipartite graphs, the result follows from Lemma 7.1. Indeed, computing a maximum weighted stable set in a bipartite graph can be done in time  $\mathcal{O}(n^3)$ , see [34].

Let  $k$  be a constant integer and  $\mathcal{C}$  a class of graphs for which there exists a polynomial time algorithm to compute maximum weighted stable sets. Let  $\mathcal{C}_k$  be the class of those graphs obtained from a graph in  $\mathcal{C}$  by adding  $k$  vertices and giving a mark to them. Then there is a polynomial time algorithm for computing a maximum weighted stable set for a graph  $G$  in  $\mathcal{C}_k$ . It suffices to try every stable subset  $S$  of the set of marked vertices, to delete all the marked vertices, to give weight zero to the neighbors of vertices of  $S$ , to run the algorithm for  $\mathcal{C}$  in what remains and to denote by  $A_S$  the stable set obtained. Then compute  $w(S \cup A_S)$ . Choose a stable set of maximum weight among the  $N \leq 2^k$  stable sets so obtained. Note that  $2^k$  is a constant.

This method works for parity-preserving extensions of complements of bipartite graphs, complements of line graphs of bipartite graphs and complements of path-cobipartite graphs. Indeed, as we show next, a graph from any of these classes cannot contain two vertex-disjoint flat paths of length at least three. So, at most one path is extended and by the remark above the desired algorithm relies on classical algorithms for maximum weighted stable set in complements of bipartite graphs, complements of line graphs of bipartite graphs, and bipartite graphs (note that maximum weighted stable set in a complement of a path-cobipartite graph corresponds to maximum weighted clique in path-cobipartite graph, and all maximal cliques of such graphs are either of size 2 or belong to the cobipartite graph obtained by removing vertices of degree 2). All this can be done in time  $\mathcal{O}(n^3)$ , see [34].

The only complement of a bipartite graph that contains a flat path of length at least 3 is  $P_4$ , line graphs of bipartite graphs cannot contain diamonds, so complements of line graphs of bipartite graphs cannot contain complements of diamonds. Hence they cannot contain two vertex-disjoint flat paths of length at least three. Now we deal with the complement  $G$  of a path-cobipartite graph. Since we know how to handle complements of bipartite graphs, we may assume that the path  $P$  from the definition is non-empty. So,  $G$  contains a vertex  $u$  of degree  $|V(G)| - 3$  (pick a vertex in  $P$ ). So,  $G$  cannot contain two disjoint flat paths of length at least 3 because the interior vertices of such paths would contradict  $\deg(u) = |V(G)| - 3$ .

To compute a maximum weighted stable set in a parity-preserving extension  $G$  of a path-cobipartite graphs  $H$ , apply the following method, where the notation  $A, B, P$  is like in the definition of path-cobipartite graphs. First observe that only vertices of  $P$  are replaced during extensions. For all stable sets  $S$  of  $G[A \cup B]$  (and there are at most  $|A| + |B| + |A||B|$  of them), consider the graph  $G_S = G \setminus (A \cup B \cup N(S))$ . Note that  $G_S$  is a bipartite graph because it is an induced subgraph of a parity-preserving extension of a path. So we can compute a maximum weighted stable set  $T_S$  of  $G_S$  in time  $\mathcal{O}(n^3)$ , see [34]. Among all stable sets  $S \cup T_S$  so constructed, choose one of maximum weight. So all this can be done in time  $\mathcal{O}(n^5)$ .

To compute a maximum weighted stable set in a parity-preserving extension  $G$  of a path-double split graph  $H$  apply the following method, where the notation  $A, B, C, D, E, k, l$  is like in the definition. First observe that only vertices of  $E$  are replaced during the extension. For all stable sets  $S$  of  $G[C \cup D]$  (and there are  $3l + 1$  of them, including  $\emptyset$ ), consider the graph  $G_S = G \setminus (C \cup D \cup N(S))$ . So,  $G_S$  is an induced subgraph of  $G \setminus (C \cup D)$  and has at most  $k$  connected components that are paths or parity-preserving extensions of paths, and hence  $G_S$  is a bipartite graph. So we can compute a maximum weighted stable set  $T_S$  of  $G_S$  in time  $\mathcal{O}(n^3)$ , see [34]. Among all stable sets  $S \cup T_S$  so constructed, choose one of maximum weight. So all this can be done in time  $\mathcal{O}(n^4)$ .

To compute a maximum weighted stable set in a parity-preserving extension  $G$  of the complement  $\overline{H}$  of a path-double split graph  $H$  there are two cases. First case, the set  $E$  is empty. Then,  $H$  is in fact a double split graph and so is  $\overline{H}$ . So  $G$  is a parity-preserving extension of a path-double split graph, and we already know how to proceed in this case. Second case, the set  $E$  is not empty. Then, all vertices in  $H$  have at least 3 non-neighbors, so in  $\overline{H}$ , no vertex has degree 2. So, no path can be extended,  $G = \overline{H}$  and to compute a maximum weighted stable set in  $G$  it suffices to compute a maximum weighted clique in  $H$ . We can do this by listing all cliques  $K$  of  $H[A \cup B \cup E]$  (including the empty set). Note that there are only linearly many such cliques. Let  $H_K$  be the subgraph of  $H$  induced by the set of all vertices  $C \cup D$  that are adjacent to all of  $K$ . It is easy to compute a maximum weighted clique  $T_K$  of  $H_K$  (it suffices to choose for each pair  $c_j, d_j$ ,  $j = 1, \dots, m$ , the vertex with bigger weight). Among all cliques  $K \cup T_K$  so constructed, choose the one of maximum weight. So all this can be done in time  $\mathcal{O}(n^4)$ .  $\square$

### 7.3. Extensions of line graphs

Extensions of line graphs are more difficult to handle than other extensions because an extension of a line graph may fail to be a line graph and a line graph may contain arbitrarily many disjoint long flat paths. Note that in this subsection, extensions are *not required to be parity-preserving*.

Let  $G'$  be a weighted graph that is an extension of a line graph  $G = L(R)$ , see Figs. 3 and 4. We now define the *transformation*  $G''$  of  $G'$ , see Fig. 5. The structure of  $G''$ , i.e. its vertices and edges, depends only on  $G$  but the weights given to its vertices depend only on  $G'$ . Let  $\mathcal{M}$  be the set of vertex-disjoint flat paths of length at least 3 of  $G$  that are extended to get  $G'$ . So,  $\mathcal{M} = \{P^1, \dots, P^k\}$  and we put  $P^i = p_1^i - \dots - p_{l_i}^i$ . For all  $1 \leq i \leq k$ , path  $P^i$  of  $G$  is replaced in  $G'$  by a set  $Q^i$  that induces either a flat claw on vertices  $q_1^i, q_2^i, q_3^i, q_4^i$  or a flat vault on vertices  $r_1^i, r_2^i, r_3^i, r_4^i, r_5^i, r_6^i$ . For all flat paths  $P^i$  of  $\mathcal{M}$ , we put  $A_2^i = N_G(p_1^i) \setminus \{p_2^i\}$ ,  $B_2^i = N_G(p_{l_i}^i) \setminus \{p_{l_i-1}^i\}$ .

For all  $1 \leq i \leq k$ , we prepare a set  $S^i$  of four new vertices  $p^i, p'^i, x^i, y^i$ . The graph  $G''$  has vertex-set:

$$V(G'') = (S^1 \cup \dots \cup S^k) \cup V(G) \setminus (P^1 \cup \dots \cup P^k).$$

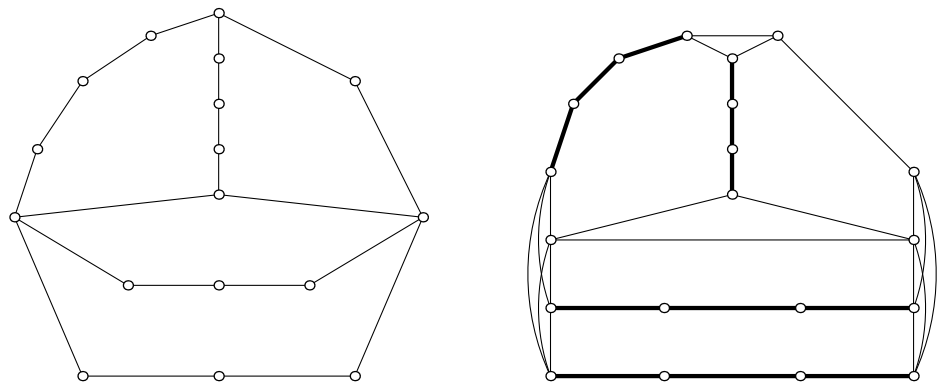


Fig. 3.  $R$  and  $L(R)$ . Paths of  $L(R)$  to be extended are represented with bold edges.

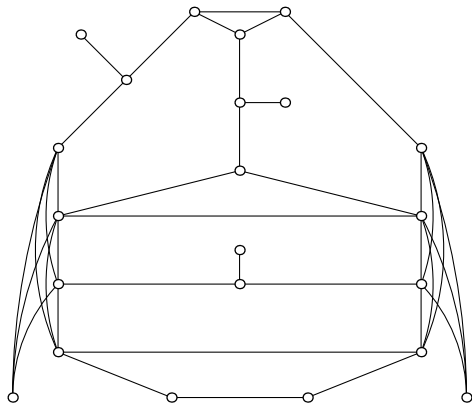


Fig. 4.  $G'$ .

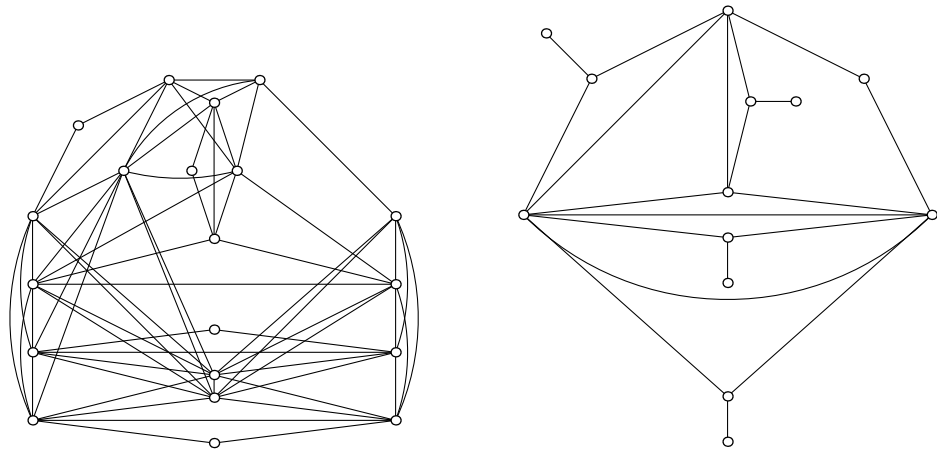


Fig. 5.  $G'' = L(R'')$  and  $R''$ .



Edges of  $G''$  depend only on edges of  $G$ . They are:

- $p^i p'^i, x^i p^i, p^i y^i, y^i p'^i, p'^i x^i, i = 1, \dots, k$ ;
- $uv$  for all  $u, v \in V(G) \cap V(G'')$  such that  $uv \in E(G)$ ;
- $p^i u$  for all  $u \in A_2^i \cap V(G''), i = 1, \dots, k$ ;
- $p'^i u$  for all  $u \in B_2^i \cap V(G''), i = 1, \dots, k$ ;
- $x^i u$  for all  $u \in (A_2^i \cup B_2^i) \cap V(G''), i = 1, \dots, k$ ;
- $p^i p^j$  for all  $i, j$  such that  $p_1^i p_1^j \in E(G)$ ;
- $p'^i p^j$  for all  $i, j$  such that  $p_{l_i}^i p_{l_j}^j \in E(G)$ ;
- $p'^i p'^j$  for all  $i, j$  such that  $p_{l_i}^i p_{l_j}^j \in E(G)$ ;
- $x^i p^j$  for all  $i, j$  such that  $p_1^i p_1^j \in E(G)$  or  $p_{l_i}^i p_1^j \in E(G)$ ;
- $x^i p'^j$  for all  $i, j$  such that  $p_1^i p_{l_j}^j \in E(G)$  or  $p_{l_i}^i p_{l_j}^j \in E(G)$ ;
- $x^i x^j$  for all  $i, j$  such that  $p_1^i p_1^j \in E(G)$  or  $p_1^i p_{l_j}^j \in E(G)$  or  $p_{l_i}^i p_1^j \in E(G)$  or  $p_{l_i}^i p_{l_j}^j \in E(G)$ .

We define the following numbers that depend only on  $G'$ :

- $a^i = \alpha(G'[\{q_1^i, q_2^i, q_4^i\}])$  for all  $i$  such that  $Q^i$  is a flat claw of  $G'$ ;
- $a^i = \alpha(G'[\{r_1^i, r_3^i, r_4^i, r_5^i\}])$  for all  $i$  such that  $Q^i$  is a flat vault of  $G'$ ;
- $b^i = \alpha(G'[\{q_2^i, q_3^i, q_4^i\}])$  for all  $i$  such that  $Q^i$  is a flat claw of  $G'$ ;
- $b^i = \alpha(G'[\{r_2^i, r_3^i, r_4^i, r_6^i\}])$  for all  $i$  such that  $Q^i$  is a flat vault of  $G'$ ;
- $c^i = \alpha(G'[\{q_2^i, q_4^i\}])$  for all  $i$  such that  $Q^i$  is a flat claw of  $G'$ ;
- $c^i = \alpha(G'[\{r_3^i, r_4^i\}])$  for all  $i$  such that  $Q^i$  is a flat vault of  $G'$ ;
- $d^i = \alpha(G'[\{q_1^i, q_2^i, q_3^i, q_4^i\}])$  for all  $i$  such that  $Q^i$  is a flat claw of  $G'$ ;
- $d^i = \alpha(G'[\{r_1^i, r_2^i, r_3^i, r_4^i, r_5^i, r_6^i\}])$  for all  $i$  such that  $Q^i$  is a flat vault of  $G'$ .

Note that from the definitions,  $c^i \leq a^i, b^i \leq d^i$  for all  $i = 1, \dots, k$ . We give the following weights to the vertices of  $G''$  (they depend on the weights in  $G'$ ):

- $w_{G''}(u) = w_{G'}(u)$  for all  $u \in V(G) \cap V(G'')$ ;
- $w_{G''}(p^i) = a^i, i = 1, \dots, k$ ;
- $w_{G''}(p'^i) = b^i, i = 1, \dots, k$ ;
- $w_{G''}(y^i) = c^i, i = 1, \dots, k$ ;
- $w_{G''}(x^i) = d^i - c^i, i = 1, \dots, k$ .

A *multigraph* is a graph where multiple edges between vertices are allowed (but we do not allow loops).

**Lemma 7.3.**  $G''$  is the line graph of a multigraph.

**Proof.** Path  $P^i$  of  $G$  corresponds to a path  $R^i = r_1^i - \dots - r_{l_i+1}^i$  of  $R$ . For all  $i = 1, \dots, k$ , path  $R^i$  is induced and has interior vertices of degree 2 in  $R$  because  $P^i$  is a flat path of  $G$ . Since paths of  $\mathcal{M}$  are vertex-disjoint, paths  $R^1, \dots, R^k$  are edge-disjoint (but they may share end-vertices). Now let us build a multigraph  $R''$  from  $R$ , see Fig. 5. We delete the interior vertices of all  $R^i$ 's. For each  $R^i$ , we add two vertices  $u^i, v^i$  and the edges  $r_1^i r_{l_i+1}^i, u^i r_1^i, u^i r_{l_i+1}^i$  and  $u^i v^i$ .

It is a routine matter to check that  $L(R'')$  is isomorphic to  $G''$ . Edge  $r_1^i r_{l_i+1}^i$  corresponds to vertex  $x^i$ , edge  $u^i r_1^i$  corresponds to vertex  $p^i$ , edge  $u^i r_{l_i+1}^i$  corresponds to  $p'^i$  and edge  $u^i v^i$  corresponds to vertex  $y^i$ . Note that possibly, two paths  $R^i$  and  $R^j$  have the same ends. For instance  $r_1^i = r_1^j$  and

$r_{i+1}^i = r_{i+1}^j$  is possible. Then, the edge  $r_1^i r_{i+1}^i$  is added twice. This is why we need  $R''$  to be a multigraph.  $\square$

**Lemma 7.4.**  $\alpha(G'') = \alpha(G')$ .

**Proof.** Let  $S'$  be a stable set of maximum weight in  $G'$ . Let us build a stable set  $S''$  of  $G''$  of same weight. In  $S''$ , we keep all vertices of  $S' \cap V(G'')$ . For all  $i$  such that  $Q^i$  is a flat claw of  $G'$ ,  $S'$  satisfies one of (i), (ii), (iii) or (iv) of Lemma 6.5. So, respective to these cases we put one of  $\{p^i\}$ ,  $\{p'^i\}$ ,  $\{y^i\}$  or  $\{x^i, y^i\}$  in  $S''$ . For all  $i$  such that  $Q^i$  is a flat vault of  $G'$ ,  $S'$  satisfies one of (i), (ii), (iii) or (iv) of Lemma 6.7. So, respective to these cases we put one of  $\{p^i\}$ ,  $\{p'^i\}$ ,  $\{y^i\}$  or  $\{x^i, y^i\}$  in  $S''$ . This yields a stable set of  $G''$  of the same weight as  $S'$ .

Conversely, if  $S''$  is a stable set of  $G''$  of maximum weight then we may assume for all  $i$ ,  $S'' \cap \{p^i, p'^i, x^i, y^i\}$  is one of  $\{p^i\}$ ,  $\{p'^i\}$ ,  $\{y^i\}$  or  $\{x^i, y^i\}$ . The only exception could be when  $w_{G''}(y^i) = 0$  and  $S'' \cap \{p^i, p'^i, x^i, y^i\} = \emptyset$  or  $\{x^i\}$ , but then we add  $y^i$  to  $S''$ . If  $Q^i$  is a flat claw, respective to these cases, we put in  $S'$  a maximum weighted stable set of one of  $G'[\{q_1^i, q_2^i, q_4^i\}]$ ,  $G'[\{q_2^i, q_3^i, q_4^i\}]$ ,  $G'[\{q_2^i, q_4^i\}]$  or  $G'[\{q_1^i, q_2^i, q_3^i, q_4^i\}]$ . If  $Q_i$  is a flat vault, respective to these cases, we put in  $S'$  a maximum weighted stable set of one of  $G'[\{r_1^i, r_3^i, r_4^i, r_5^i\}]$ ,  $G'[\{r_2^i, r_3^i, r_4^i, r_6^i\}]$ ,  $G'[\{r_3^i, r_4^i\}]$  or  $G'[\{r_1^i, r_2^i, r_3^i, r_4^i, r_5^i, r_6^i\}]$ . This yields a stable set of  $G'$  of the same weight as  $S''$ .  $\square$

**Lemma 7.5.** *There is an algorithm with the following specification:*

**Input:** A weighted graph  $G'$  that is an extension of a line graph  $G$ .

**Output:** A maximum weighted stable set of  $G'$ .

**Running time:**  $\mathcal{O}(n^3)$ .

**Proof.** Build the transformation  $G''$  of  $G'$  as explained above. So, by Lemma 7.3,  $G''$  is the line graph of a multigraph. Compute a multigraph  $R$  such that  $G'' = L(R)$  (see [27,33]), then compute in  $R$  a matching of maximum weight by Edmonds' algorithm (see [34,20]). It corresponds to a maximum weighted stable set in  $G''$ . By Lemma 7.4, this maximum weighted stable set has the same weight as a maximum weighted stable set  $S'$  of  $G'$ . Note that the proof of Lemma 7.4 shows how to actually obtain  $S'$ .  $\square$

#### 7.4. Extensions of basic even-hole-free graphs

Here again, extensions are *not required to be parity-preserving*.

**Lemma 7.6.** *There is an algorithm with the following specification:*

**Input:** A weighted graph  $G$  that is an extension of a graph from  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ .

**Output:** A maximum weighted stable set and a maximum weighted clique of  $G$ .

**Running time:**  $\mathcal{O}(n^4)$ .

**Proof.** Let  $H$  be a graph in  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$  and  $\mathcal{M}$  a set of vertex-disjoint flat paths of  $H$  of length at least 3, such that the input graph  $G$  is an extension of  $(H, \mathcal{M})$ . Let  $A$  be a set of vertices of  $H$  such that  $|A| \leq 2$  and  $H \setminus A$  is a line graph (it takes time  $\mathcal{O}(n^4)$  to find  $A$ , by checking for all possible pairs whether their removal yields a line graph). Let  $\mathcal{M}'$  be the set of paths of  $\mathcal{M}$  that contain some vertex of  $A$ . Note that  $|\mathcal{M}'| \leq 2$ .

Since  $|A| \leq 2$  and  $|\mathcal{M}'| \leq 2$ , there is a set of vertices  $B$  such that  $|B| \leq 12$  and  $G$  is obtained from  $G'$  by adding vertices of  $B$  where  $G'$  is an extension of  $(H \setminus A, \mathcal{M} \setminus \mathcal{M}')$ . By Lemma 7.5, a maximum weighted stable set of  $G'$  can be computed in time  $\mathcal{O}(n^3)$ . To compute a maximum weighted stable set of  $G$  it suffices to try every stable set  $S$  of  $B$ , to delete all vertices of  $S \setminus B$ , to give weight zero to neighbors of vertices of  $S$ , and to compute the maximum weighted stable set of the remaining

graph (which is as we noted an extension of a line graph). Choose a maximum weighted stable set so obtained. Clearly all this can be done in time  $\mathcal{O}(n^3)$ .

To compute a maximum weighted clique, it suffices to notice that  $G$  contains linearly many inclusion-wise maximal cliques. So, it suffices to list them and to choose one of maximum weight.  $\square$

It is convenient to sum up all the results of the section:

**Lemma 7.7.** *There is an algorithm with the following specification:*

**Input:** A weighted graph  $G$  that is a parity preserving extension of a graph from  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$  or any extension of a graph from  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ .

**Output:** A maximum weighted stable set and a maximum weighted clique of  $G$ .

**Running time:**  $\mathcal{O}(n^5)$ .

**Proof.** Follows from Lemmas 7.2, 7.5 and 7.6 for computing stable sets. For cliques, it is done in Lemma 7.6 for graphs from  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ . For  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$ , it suffices to notice that the class is self-complementary, so we may rely on the algorithm for stable sets.  $\square$

## 8. Constructing the decomposition tree

We now give algorithms to construct several decomposition trees for graphs in our classes. First we show how to build a decomposition tree with the usual parity preserving blocks (as defined in Section 3). Then we show how to reprocess such a tree to get a tree with clique-blocks, even blocks or odd blocks according to what we need to optimize.

### 8.1. Tree with parity preserving blocks

We define now a decomposition tree  $T_G$  of a graph  $G \in \mathcal{D}$ , where  $\mathcal{D}$  is one of  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$ ,  $\mathcal{C}_{\text{BASIC}}^{\text{NO CUTSET}}$ ,  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ . We call  $\mathcal{D}_{\text{BASIC}}$  the class of all basic graphs associated to the class (so, if  $\mathcal{D} = \mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$  then  $\mathcal{D}_{\text{BASIC}} = \mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$  and if  $\mathcal{D} = \mathcal{C}_{\text{BASIC}}^{\text{EHF}}$  then  $\mathcal{D}_{\text{BASIC}} = \mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ ).

We decompose a graph  $G \in \mathcal{D}$  using extreme 2-joins into basic graphs. Let us now define more precisely what we call decomposition tree (proving its existence and constructing it will be done later).

*Decomposition tree  $T_G$  of depth  $p \geq 1$  of a graph  $G \in \mathcal{D}$  that has a connected non-path 2-join.*

- (i) The root of  $T_G$  is  $(G^0, \mathcal{M}^0)$ , where  $G^0 = G$  and  $\mathcal{M}^0 = \emptyset$ .
- (ii) Each node of the decomposition tree is a pair  $(H, \mathcal{M})$  where  $H$  is a graph of  $\mathcal{D}$  and  $\mathcal{M}$  is a set of disjoint flat paths of length 3 or 4 of  $H$ .  
The non-leaf nodes of  $T_G$  are pairs  $(G^0, \mathcal{M}^0), \dots, (G^{p-1}, \mathcal{M}^{p-1})$ . Each non-leaf node  $(G^i, \mathcal{M}^i)$  has two children. One is  $(G^{i+1}, \mathcal{M}^{i+1})$ , the other one is  $(G_B^{i+1}, \mathcal{M}_B^{i+1})$ .  
The leaf-nodes of  $T_G$  are the pairs  $(G_B^1, \mathcal{M}_B^1), \dots, (G_B^p, \mathcal{M}_B^p)$  and  $(G^p, \mathcal{M}^p)$ . Graphs  $G_B^1, \dots, G_B^p$  all belong to  $\mathcal{D}_{\text{BASIC}}$ .
- (iii) For  $i = 0, \dots, p-1$ ,  $G^i$  has a connected non-path 2-join  $(X_1^i, X_2^i)$  that is extreme with extreme side  $X_1^i$  and that is  $\mathcal{M}^i$ -independent. Graphs  $G^{i+1}$  and  $G_B^{i+1}$  are the parity preserving blocks of  $G^i$  w.r.t.  $(X_1^i, X_2^i)$  (as defined in Section 3), whose marker paths are of length 3 or 4. The block  $G_B^{i+1}$  corresponds to the extreme side  $X_1^i$ , i.e.  $X_1^i \subseteq V(G_B^{i+1})$ .  
Set  $\mathcal{M}_B^{i+1}$  consists of paths from  $\mathcal{M}^i$  whose vertices are in  $X_1^i$ . Note that the marker path used to construct the block  $G_B^{i+1}$  does not belong to  $\mathcal{M}_B^{i+1}$ .  
Set  $\mathcal{M}^{i+1}$  consists of paths from  $\mathcal{M}^i$  whose vertices are in  $X_2^i$  together with the marker path  $p^{i+1}$  used to build  $G^{i+1}$ .

(iv)  $\mathcal{M}_B^1 \cup \dots \cup \mathcal{M}_B^p \cup \mathcal{M}^p$  is the set of all marker paths used in the construction of the nodes  $G^1, \dots, G^p$  of  $T_G$ , and sets  $\mathcal{M}_B^1, \dots, \mathcal{M}_B^p, \mathcal{M}^p$  are pairwise disjoint.

Node  $(G^p, \mathcal{M}^p)$  is a leaf of  $T_G$  and is called the *deepest node* of  $T_G$ . Note that all leaves of  $T_G$  except possibly the deepest node are basic.

**Lemma 8.1.** *For any decomposition tree  $T_G$ , the depth of  $T_G$  is at most  $n$ .*

**Proof.** Let a *branch* of a graph be any path of length at least 2 whose endvertices are both of degree at least 3 and whose interior vertices are of degree 2 (in the graph). For a graph  $G$ , let  $\nu(G)$  be the number of vertices of degree at least 3 in  $G$ , and  $\tau(G)$  the number of branches in  $G$ . We will show that for  $i = 0, \dots, p - 1$ :

$$(1) \quad \nu(G^{i+1}) + \tau(G^{i+1}) < \nu(G^i) + \tau(G^i).$$

This implies the lemma because it shows that  $p$  is at most

$$\nu(G) + \tau(G) \leq n.$$

Let  $i \in \{0, \dots, p - 1\}$  and let  $(X_1^i, X_2^i, A_1^i, B_1^i, A_2^i, B_2^i)$  be a split of the 2-join  $(X_1^i, X_2^i)$ . Let  $P^{i+1} = a_1^i - \dots - b_1^i$  be the marker path of  $G^{i+1}$ . Let  $P = u - \dots - u'$  be a path of  $G^i$  such that  $u \in A_1^i, u' \in B_1^i$  and  $P \setminus \{u, u'\} \subseteq X_1^i \setminus (A_1^i \cup B_1^i)$  (note that such a path exists since  $(X_1^i, X_2^i)$  is connected). We choose such a path  $P$  with a minimum number of vertices of degree at least 3. Since  $(X_1^i, X_2^i)$  is a non-path 2-join, there is a vertex  $q \in X_1^i \setminus P$ . Observe that  $d_{G^{i+1}}(a_1^i) \leq d_{G^i}(u)$  and  $d_{G^{i+1}}(b_1^i) \leq d_{G^i}(u')$ , and hence  $\nu(G^{i+1}) \leq \nu(G^i)$ . When  $q$  is of degree at least 3 we have  $\nu(G^{i+1}) < \nu(G^i)$ . Also, since exactly one branch of  $G^{i+1}$  intersects  $P^{i+1}$ , we have  $\tau(G^{i+1}) \leq \tau(G^i) + 1$ .

First suppose that  $P$  contains a vertex of degree 2. Then there is a branch  $P^*$  of  $G^i$  that contains a node of  $P$ , and hence  $\tau(G^{i+1}) \leq \tau(G^i)$ . If  $d_{G^i}(q) \geq 3$ , then  $\nu(G^{i+1}) < \nu(G^i)$ , and hence (1) holds. So suppose that  $d_{G^i}(q) = 2$ . Then  $q$  belongs to a branch  $Q^*$  of  $G^i$ . Clearly  $P^* \neq Q^*$ , so  $\tau(G^{i+1}) < \tau(G^i)$ , and hence (1) holds.

Now we may assume that all vertices of  $P$  are of degree at least 3. Suppose that  $P$  is of length at least 2. So  $\nu(G^{i+1}) \leq \nu(G^i) - 1$ . If  $d_{G^i}(q) \geq 3$  then  $\nu(G^{i+1}) \leq \nu(G^i) - 2$ , and hence (1) holds. Otherwise,  $q$  belongs to a branch of  $G^i$  and so  $\tau(G^{i+1}) \leq \tau(G^i)$ , and hence (1) holds.

Finally we assume that  $P$  is of length 1, and both  $u$  and  $u'$  are of degree at least 3. By Lemma 3.3 and Lemma 3.2(iv) and (v),  $|A_1^i|, |B_1^i| \geq 2$ . Since  $A_2^i \cup \{u, u'\}$  is not a star cutset of  $G^i$ , there is a path  $T = t - \dots - t'$  such that  $t \in A_1^i \setminus \{u\}, t' \in B_1^i \setminus \{u'\}$  and  $T \setminus \{t, t'\} \subseteq X_1^i \setminus (A_1^i \cup B_1^i)$ . By the choice of  $P$ ,  $T$  contains at least two vertices of degree at least 3. So,  $\nu(G^{i+1}) \leq \nu(G^i) - 2$  and hence (1) holds.  $\square$

**Lemma 8.2.** *There is an algorithm with the following specification:*

**Input:** A graph  $G$  in  $\mathcal{D}$  that has a connected non-path 2-join.

**Output:** A decomposition tree  $T_G$  of  $G$  of depth at most  $n$  whose leaves are all in  $\mathcal{D}_{\text{BASIC}}$ .

**Running time:**  $\mathcal{O}(n^4m)$ .

**Proof.** Let the root of  $T_G$  be  $(G^0, \mathcal{M}^0) = (G, \emptyset)$ . We suppose by induction that a decomposition tree of depth  $i$  has been constructed. So, the deepest leaf  $(G^i, \mathcal{M}^i)$  is a pair such that  $G^i \in \mathcal{D}$  and  $\mathcal{M}^i$  is a set of vertex-disjoint flat paths of  $G^i$  of length 3 or 4. Apply the algorithm from Lemma 4.6 to  $G^i$  and  $\mathcal{M}^i$ . One of the following is the output of this algorithm.

**Case 1.** An extreme connected non-path 2-join  $(X_1^i, X_2^i)$  of  $G^i$ , with say  $X_1^i$  being the extreme side, that is  $\mathcal{M}^i$ -independent.

Let  $G^{i+1}$  and  $G_B^{i+1}$  be parity-preserving blocks of decomposition of  $G^i$  w.r.t.  $(X_1^i, X_2^i)$  (as defined in Section 3), whose marker paths are of length 3 or 4, and block  $G_B^{i+1}$  corresponds to  $X_1^i$ -side.

By Lemmas 3.8 and 3.11,  $G^{i+1}$  and  $G_B^{i+1}$  are both in  $\mathcal{D}$ . Since  $G_B^{i+1}$  is the block that corresponds to  $X_1^i$ ,  $G_B^{i+1}$  has no connected non-path 2-join. If  $\mathcal{D} = \mathcal{C}_{\text{NO SC}}^{\text{EHF}}$  then by Theorem 2.1,  $G_B^{i+1} \in \mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ . If  $\mathcal{D} = \mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$ , then by Theorem 2.6,  $G_B^{i+1} \in \mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$ . Therefore,  $G_B^{i+1} \in \mathcal{D}_{\text{BASIC}}$ .

Since  $(X_1^i, X_2^i)$  is  $\mathcal{M}^i$ -independent, for  $P \in \mathcal{M}^i$ , either  $P \subseteq X_1^i$  or  $P \subseteq X_2^i$ . Let  $\mathcal{M}_B^{i+1}$  be the set of paths of  $\mathcal{M}^i$  that belong to  $X_1^i$ . Let  $\mathcal{M}^{i+1}$  be the set of paths of  $\mathcal{M}^i$  that belong to  $X_2^i$  together with the marker path  $p^{i+1}$  of  $G^{i+1}$ . Clearly  $\mathcal{M}_B^{i+1}$  (resp.  $\mathcal{M}^{i+1}$ ) is a set of vertex-disjoint flat paths of  $G_B^{i+1}$  (resp.  $G^{i+1}$ ) of length 3 or 4,  $\mathcal{M}_B^{i+1} \cap \mathcal{M}^{i+1} = \emptyset$  and  $\mathcal{M}_B^{i+1} \cup \mathcal{M}^{i+1} = \mathcal{M}^i \cup \{p^{i+1}\}$ .

Hence, in Case 1, we have built a deeper decomposition tree of  $G$ .

**Case 2.**  $G^i$  or a block of decomposition of  $G^i$  w.r.t. some 2-join whose marker path is of length at least 3, has a star cutset.

Since  $G^i \in \mathcal{D}$  and by Lemma 3.3,  $G^i$  cannot have a star cutset. By Lemma 3.2(ii), any 2-join of  $G^i$  is connected, and hence by Lemma 3.7 this case actually cannot happen.

**Case 3.**  $G^i$  has no connected non-path 2-join.

Note that  $i \geq 1$  since  $G$  is assumed to have a non-path connected 2-join. If  $\mathcal{D} = \mathcal{C}_{\text{NO SC}}^{\text{EHF}}$  then by Theorem 2.1,  $G^i \in \mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ . If  $\mathcal{D} = \mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$ , then by Theorem 2.6,  $G^i \in \mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$ . Therefore  $G^i \in \mathcal{D}_{\text{BASIC}}$ .

By Lemma 8.1, we see that Case 3 must happen at some point, after at most  $n$  iterations. So, when Case 3 happens we output the tree  $T_G$  and stop. All the leaves of  $T_G$  are basic. Since the complexity of the algorithm from Lemma 4.6 is  $\mathcal{O}(n^3m)$  and there are at most  $n$  iterations, the algorithm for constructing  $T_G$  runs in time  $\mathcal{O}(n^4m)$ .  $\square$

## 8.2. Clique-decomposition tree

The *clique-decomposition tree* is used to compute maximum cliques for graphs in  $\mathcal{D}$ . This tree  $T_G^C$  has the same definition as  $T_G$  except that weights are given to the vertices. Let us be more precise by defining how to compute the children of  $(G^i, \mathcal{M}^i)$  in  $T_G^C$ . Recall that the graph  $G^i$  has an extreme 2-join  $(X_1^i, X_2^i)$  with extreme side  $X_1^i$ . Its children are  $(G_B^{i+1}, \mathcal{M}_B^{i+1})$  and  $(G^{i+1}, \mathcal{M}^{i+1})$ . In  $G_B^{i+1}$ , all vertices from  $G[X_1^i]$  keep their weights (as they are in  $G^i$ ) and vertices of the new marker path receive weight 0. The weights of  $G^{i+1}$  are assigned as in the construction of the clique block in Section 5. Note that computing the weights in the construction of  $G^{i+1}$  relies on several computations on  $G^i[X_1^i]$ , or equivalently on  $G_B^{i+1}$  which is a basic graph. So by Lemma 7.7, building  $G^{i+1}$  takes time  $\mathcal{O}(n^5)$ . This construction leads to the following result:

**Lemma 8.3.** *There is an algorithm with the following specification:*

**Input:** A weighted graph  $G$  that is in  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$  or in  $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$  and that has a connected non-path 2-join.

**Output:** A maximum weighted clique of  $G$ .

**Running time:**  $\mathcal{O}(n^6)$ .

**Proof.** By Lemma 8.2, we build a decomposition tree for  $G$ , and as explained above we reprocess the tree to get a clique-decomposition tree. By repeatedly applying Lemma 5.1, we see that  $\omega(G^0) = \omega(G^1) = \dots = \omega(G^p)$ . We can compute a maximum weighted clique in the basic graph  $G^p$  by Lemma 7.7, and the proof of Lemma 5.1 shows how to backtrack such a maximum weighted clique to  $G$ .  $\square$

Note that for graphs of  $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$ , the lemma above is not so interesting because a faster algorithm exists for the class  $\mathcal{C}^{\text{EHF}}$ , see the introduction.

### 8.3. Stable-decomposition tree for $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$

We define now the *stable-decomposition tree*  $T_G^S$  of a graph in  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$ . The tree  $T_G^S$  has the same definition as  $T_G$  except that even or odd blocks are used sometimes and that the sets  $\mathcal{M}^i$ 's and  $\mathcal{M}_B^i$ 's will be sets of disjoint flat claws and vaults (instead of paths). Let us be more precise.

*Decomposition tree  $T_G^S$  of depth  $p \geq 1$  of a weighted graph  $G \in \mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$  that has a connected non-path 2-join.*

- (i) The root of  $T_G^S$  is  $(G''^0, \mathcal{M}''^0)$ , where  $G''^0 = G$  and  $\mathcal{M}''^0 = \emptyset$ .
- (ii) For each node  $(H, \mathcal{M})$  of the decomposition tree,  $H$  is a Berge graph and  $\mathcal{M}$  is a set of disjoint flat claws or flat vaults of  $H$ .  
The non-leaf nodes of  $T_G^S$  are pairs  $(G''^0, \mathcal{M}''^0), \dots, (G''^{p-1}, \mathcal{M}''^{p-1})$ . Each non-leaf node  $(G''^i, \mathcal{M}''^i)$  has two children. One is  $(G''^{i+1}, \mathcal{M}''^{i+1})$ , the other one is  $(G_B''^{i+1}, \mathcal{M}_B''^{i+1})$ .  
The leaf-nodes of  $T_G^S$  are the pairs  $(G_B''^1, \mathcal{M}_B''^1), \dots, (G_B''^p, \mathcal{M}_B''^p)$  and  $(G''^p, \mathcal{M}''^p)$ . Graphs  $G_B''^1, \dots, G_B''^p$  are all parity-preserving extensions of graphs from  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$ .
- (iii) For  $i = 0, \dots, p-1$ ,  $G''^i$  has a connected non-path 2-join  $(X_1''^i, X_2''^i)$ .  
Graph  $G_B''^{i+1}$  is the parity-preserving block of  $G^i$  w.r.t.  $(X_1''^i, X_2''^i)$  (as defined in Section 3), whose marker path is of length 3 or 4 and which corresponds to the side  $X_1''^i$ , i.e.  $X_1''^i \subseteq V(G_B''^{i+1})$ . Vertices of  $X_1''^i$  keep their weight from  $G''^i$  and vertices of the marker path receive weight zero. Graph  $G''^{i+1}$  is the even block or the odd block of  $G''^i$  w.r.t.  $(X_1''^i, X_2''^i)$ , according to the  $X_1''^i$ -parity of  $(X_1''^i, X_2''^i)$ .  
Set  $\mathcal{M}_B''^{i+1}$  consists of claws and vaults from  $\mathcal{M}''^i$  whose vertices are in  $X_1''^i$ . Note that the marker path used to construct the block  $G_B''^{i+1}$  does not belong to  $\mathcal{M}_B''^{i+1}$ .  
Set  $\mathcal{M}''^{i+1}$  consists of claws and vaults from  $\mathcal{M}''^i$  whose vertices are in  $X_2''^i$  together with the claw or the vault  $P''^{i+1}$  used to build  $G''^{i+1}$ .
- (iv)  $\mathcal{M}_B''^1 \cup \dots \cup \mathcal{M}_B''^p \cup \mathcal{M}''^p$  is the set of all marker claws or vaults used in the construction of the nodes  $G''^1, \dots, G''^p$  of  $T_G^S$ , and sets  $\mathcal{M}_B''^1, \dots, \mathcal{M}_B''^p, \mathcal{M}''^p$  are pairwise disjoint.

The existence of  $T_G^S$  is not clear since introducing even and odd blocks may create star cutsets (and so balanced skew partitions) in our graphs, so that we cannot rely on Theorem 2.6 to build the tree recursively. But here we show how to actually construct  $T_G^S$  by reprocessing  $T_G$ .

Start from  $T_G$  and for each node  $(G^i, \mathcal{M}^i)$ ,  $i = 1, \dots, p$  of  $T_G$ , extend the marker path introduced in that node to obtain a graph  $G''^i$ . Accordingly, replace the marker paths in the graphs  $G_B^i$  and sets  $\mathcal{M}_B^i, \mathcal{M}^i$  by marker claws and vaults to obtain  $G_B''^i, \mathcal{M}_B''^i$  and  $\mathcal{M}''^i$ . We obtain the nodes  $(G''^i, \mathcal{M}''^i)$  and  $(G_B''^i, \mathcal{M}_B''^i)$  of  $T_G^S$ .

Note that extending a flat path  $P$  of length at least 3 in a graph  $H$  with a  $\{P\}$ -independent connected 2-join  $(X_1, X_2)$ , yields a graph  $H''$  that has a connected 2-join  $(X_1'', X_2'')$  naturally arising from  $(X_1, X_2)$ : put all vertices of  $X_1 \setminus P$  in  $X_1''$ , all vertices of  $X_2 \setminus P$  in  $X_2''$  and put the claw or the vault arising from  $P$  in  $X_1''$  when  $P \subseteq X_1$  and in  $X_2''$  when  $P \subseteq X_2$ . So, the connected 2-joins  $(X_1''^i, X_2''^i)$ 's all exist and are immediate to find from  $T_G$ .

Note that by Lemma 6.9, all nodes of  $T_G^S$  are in  $\mathcal{C}_{\text{BERGE}}$ . So, all the 2-joins  $(X_1''^i, X_2''^i)$ 's are either  $X_1''^i$ -even or  $X_1''^i$ -odd and we can choose whether we use an even or an odd block.

Note that computing the weights in the construction of the even or odd block  $G''^{i+1}$  relies on several computations on  $G''^i[X_1'']$ , or equivalently on  $G_B''^{i+1}$  which is a parity-preserving extension of a graph from  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$ , so the computations can be done in time  $O(n^5)$  by Lemma 7.7.

### 8.4. Stable-decomposition tree for $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$

For an even-hole-free graph the tree  $T_G^S$  as defined above may fail to exist. Because building an odd block does not preserve being even-hole-free. So, 2-joins appearing in the decomposition tree

may fail to be either  $X_1$ -even or  $X_1$ -odd, so we do not know when to use even or odd blocks. But with a little twist, we can define a useful tree.

The definition of  $T_G^S$  for a graph in  $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$  is very similar to the definition for  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$ , so we do not repeat it and point out the differences instead.

The main difference is when to use odd or even block. Recall that  $G''^i$  has a 2-join  $(X_1''^i, X_2''^i)$ . Let  $(X_1''^i, X_2''^i, A_1''^i, B_1''^i, A_2''^i, B_2''^i)$  be a split of this 2-join. We define  $a^i = \alpha(G[A_1''^i \cup C_1''^i])$ ,  $b^i = \alpha(G[B_1''^i \cup C_1''^i])$ ,  $c^i = \alpha(G[C_1''^i])$  and  $d^i = \alpha(G[X_1''^i])$ . If  $a^i + b^i \leq c^i + d^i$  then  $G''^{i+1}$  is the even block  $G''^i$  w.r.t.  $(X_1''^i, X_2''^i)$ . Else, it is the odd block. Note that graphs in  $T_G^S$  are not required to be in  $\mathcal{C}^{\text{EHF}}$ .

The other difference is that graphs  $G_B''^1, \dots, G_B''^p$  and  $G''^p$  are any (and not only parity-preserving) extensions of graphs from  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ . Note that in Lemma 7.7, the extensions are not required to be parity-preserving for  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$ .

**Lemma 8.4.** *There is an algorithm with the following specification:*

**Input:** A weighted graph  $G$  that is in  $\mathcal{C}_{\text{NO CUTSET}}^{\text{BERGE}}$  or in  $\mathcal{C}_{\text{NO SC}}^{\text{EHF}}$  and that has a connected non-path 2-join.

**Output:** A maximum weighted stable set of  $G$ .

**Running time:**  $\mathcal{O}(n^4 m)$ .

**Proof.** By Lemma 8.2, we build a decomposition tree for  $G$ , and as explained above we reprocess the tree to get a stable-decomposition tree  $T_G^S$ . By repeatedly applying Lemmas 6.6 and 6.8, we see that  $\alpha(G''^0) = \alpha(G''^1) = \dots = \alpha(G''^p)$ . Note that the inequalities necessary to apply Lemmas 6.6 and 6.8 are satisfied. For Berge graphs, this follows from the fact that all nodes of  $T_G^S$  are Berge so that we can rely on Lemmas 6.3 and 6.4. For even-hole-free graphs, the inequalities are true from the definition of  $T_G^S$ .

We can compute a maximum weighted stable set in the extension of basic graph  $G''^p$  by Lemma 7.7 (when  $G$  is Berge, the extension is parity-preserving), and the proofs of Lemmas 6.6 and 6.8 show how to backtrack such a maximum weighted stable set to  $G$ .  $\square$

## 9. Optimization algorithms

**Theorem 9.1.** *There is an algorithm with the following specification:*

**Input:** A weighted graph  $G$  that is either a Berge graph with no balanced skew partition, no connected non-path 2-join in the complement and no homogeneous pair; or an even-hole-free graph with no star cutset.

**Output:** A maximum weighted stable set and a maximum weighted clique of  $G$ .

**Running time:**  $\mathcal{O}(n^6)$ .

**Proof.** If  $G$  is in  $\mathcal{C}_{\text{BASIC}}^{\text{BERGE}}$  or in  $\mathcal{C}_{\text{BASIC}}^{\text{EHF}}$  we rely on Lemma 7.7. Else, by Theorem 2.1 or 2.6,  $G$  has a connected non-path 2-join. So, we may rely on Lemmas 8.3 and 8.4.  $\square$

Let us now point out that our method works for our two classes for different reasons. For Berge graphs, it is because our even and odd blocks are class-preserving. For even-hole-free graphs, it is because the basic class is restricted to graphs obtained from line graphs by adding a bounded number of vertices. In fact, our method works for something more general than even-hole-free graphs with no star cutset. Let  $L_k$  be the class of graphs obtained from line graphs by adding  $k$  vertices. If  $k$  is fixed, we can compute maximum weighted cliques and stable sets for any class that is decomposable with extreme 2-joins into graphs of  $L_k$ . For  $k=0$ , this gives a subclass of claw-free graphs. For  $k=2$  this gives a super-class of even-hole-free graphs with no star cutset.

**Theorem 9.2.** *There exists an algorithm of complexity  $\mathcal{O}(n^7)$  whose input is a Berge graph  $G$  with no balanced skew partition, no connected non-path 2-join in the complement and no homogeneous pair, and whose output is an optimal coloring of  $G$  and an optimal coloring of  $\bar{G}$ .*

**Proof.** There exists a combinatorial coloring algorithm for an input perfect graph  $G$  that uses at most  $n$  times as subroutines algorithms for maximum cliques and stable sets. See [26] or Corollary 67.2c in [34]. This algorithm relies on the fact that perfect graphs are closed under taking induced subgraphs and replicating vertices. Our class is not, but taking induced subgraphs is easily simulated by giving weight 0 to a vertex and replicating  $k$  times a vertex is simulated by giving weight  $k$  to the vertex. The method also works for  $\bar{G}$  because we may compute maximum weighted cliques and stable sets for  $\bar{G}$  as well.  $\square$

## 10. NP-completeness

Here, we give a class  $\mathcal{C}$  of graph for which computing a maximum stable set is NP-hard. The interesting feature of class  $\mathcal{C}$  is that all graphs in  $\mathcal{C}$  are decomposable along extreme 2-joins into one bipartite graph and several gem-wheels where a *gem-wheel* is any graph made of an induced cycle of length at least 5 together with a vertex adjacent to exactly four consecutive vertices of the cycle. Note that a gem-wheel is a line graph (of a cycle with one chord). Our NP-completeness result (proved jointly with Guyslain Naves) shows that being able to decompose along extreme 2-joins is not enough in general to compute stable sets. This suggests that being in  $\mathcal{C}^{\text{PARITY}}$  is essential for computing stable sets along 2-joins and that the inequalities of Subsection 6.1 capture an essential feature of  $\mathcal{C}^{\text{PARITY}}$ .

Here, *extending* a flat path  $P = p_1 - \dots - p_k$  of a graph means deleting the interior vertices of  $P$  and adding three vertices  $x, y, z$  and the following edges:  $p_1x, xy, yp_k, zp_1, zx, zy, zp_k$ . By extending a graph  $G$  we mean extending all paths of  $\mathcal{M}$  where  $\mathcal{M}$  is a set of flat paths of length at least 3 of  $G$ . Class  $\mathcal{C}$  is the class of all graphs obtained by extending 2-connected bipartite graphs. From the definition, it is clear that all graphs of  $\mathcal{C}$  are decomposable along extreme connected non-path 2-joins. One leaf of the decomposition tree will be the underlying bipartite graph. All the others leaves will be gem-wheels.

We call *4-subdivision* any graph  $G$  obtained from a graph  $H$  by subdividing four times every edge. More precisely, every edge  $uv$  of  $H$  is replaced by an induced path  $u-a-b-c-d-v$  where  $a, b, c, d$  are of degree two. It is easy to see that  $\alpha(G) = \alpha(H) + 2|E(H)|$ . This construction, essentially due to Poljak [30], yields as observed by Guyslain Naves:

**Theorem 10.1.** (See Naves [29].) *The problem whose instance is a graph  $G$  from  $\mathcal{C}$  and an integer  $k$ , and whose question “Does  $G$  contain a stable set of size at least  $k$ ?” is NP-complete.*

**Proof.** Let  $H$  be any graph. First we subdivide 5 times every edge of  $H$ . So each edge  $ab$  is replaced by  $P_7 = a - p_1 - \dots - p_5 - b$ . The graph  $H'$  obtained is bipartite. Now we build an extension  $G$  of  $H'$  by replacing all the  $P_5$ 's  $p_1 - \dots - p_5$  arising from the subdivisions in the previous step by  $P_4$ 's. And for each  $P_4$  we add a new vertex complete to it and we call *apex vertices* all these new vertices. The graph  $G$  that we obtain is in  $\mathcal{C}$ . It is easy to see that there exists a maximum stable set of  $G$  that contain no apex vertex because an apex vertex of a maximum stable set can be replaced by one vertex of its neighborhood. So, we call  $G'$  the graph obtained from  $G$  by deleting all the apex vertices and see that  $\alpha(G') = \alpha(G)$ . Also,  $G'$  is the 4-subdivision arising from  $H$ . So from the remark above, maximum stable sets in  $H$  and  $G$  have sizes that differ by  $2|E(H)|$ .  $\square$

Our NP-completeness result is related to Lemma 6.10. With respect to computing maximum stable sets, these two results say in a sense that gem-blocks carry enough information to encode one side of a 2-join.

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